Algebraic Geometry I&II

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Contents

Part I

Algebraic Geometry I

22/8/29 to 22/10/26.

Lecture 1 to Lecture 14 and a part of Lecture 15, but actually, I put this part into Lecture 14.

1 Lecture 1.

22/8/29.

1.1 §A. What is Algebraic Geometry?

• Objects: set of solutions of finitely many polynomials in several varibles.

Example 1.1.

(1) Nodal curves: $X^3 + Y^3 - XY = 0$, see the local of the singular point is a 'cross'.

(2) Cuspidal curves: $Y^2 - X^3$, the local is a cusp.

(3) Elliptic curves: $Y^2 - X(X-1)(X-2)$, it consists of two parts.

(4) let k be $\mathbb{Q}$, consider $\mathbb{Q}^2$, the rational plane, and $X^2 - aY^2 - 1$ with $a \in \mathbb{Q}$, called **Pell equation**, and the solutions are $(\frac{t^2+a}{t^2-a}, \frac{2t}{t^2-a})$, in particular, see the circle case which we have known the rational solutions before.

• Relations:

(1) $k = \mathbb{R}$ related to differential geometry.

(2) $k = \mathbb{C}$ related to complex geometry.

(3) $k = \mathbb{Q}$, $\mathbb{F}_p$ related to number theory.

• Solve Problems

(1) Classfication(up to some equivalence).
e.g. MMP, thanks to Mori, we go to higher-dimensional geometry.

(2) Topological Properties.
e.g. Hodge theory.

(3) Existence of Solutions.($k = \mathbb{Q}$)
e.g. Diophantine geometry.

(4) Counting Problems.
e.g. Gromov-Witten theory.

1.2 §B. What is a Space?

(a) $|X|$ = set of points.

(b) $\mathcal{T}$ = topology on X.

(c) $\mathcal{O}$ = functions on open subsets of X.

(d) Local model, like Balls on Euclidean topology.

Remark 1.2. The structure of a space can be understood by considering **all** functions on **all** open subsets $\Rightarrow$ **Sheaf!**

I. Presheaves and Sheaves

1.3 §A. Presheaves and Sheaves

Let X be a topological space, $\mathcal{D}(X)$ = set of open subsets of X.
For $x \in X$, $U(x)$= set of open neighborhoods containing x.

Definition 1.3 (Presheaf)**. Presheaf** is a contravariant functor $\mathscr{F} : \mathcal{C}^{op} \to \mathcal{S}et$, more explicitly, we have restriction morphisms $\text{res}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ for opens $V \subseteq U$, satisfying additional axioms:

A1. $\mathcal{O}(\varnothing)$ is the terminal object in the target category.

A2. $\text{res}_{U,U} = \text{id}_{\mathscr{F}(U)}$.

A3. $\text{res}_{U,W} = \text{res}_{V,W} \circ \text{res}_{U,V}$.

We will use the special case $\mathfrak{Ab}$.

Example 1.4.

(1) Zero sheaf: $\mathscr{F}(U) = 0$ for all open subset U.

(2) Constant presheaf: take a fixed abelian group A, $\mathscr{F}(U) = A$ for all open subset U.

(3) Presheaf of continuous functions: take $U \in \mathcal{D}(X)$,

$$\mathscr{F}(U) = \begin{cases} \{f : U \to \mathbb{R} \text{ continuous } \} & U \neq \varnothing, \\ 0 & U = \varnothing. \end{cases}$$

(4) Presheaf of holomorphic functions: take $X = \mathbb{C}^n$, $\mathscr{F}(U)$ is set of $f : U \to \mathbb{C}$ which are holomorphic.

Definition 1.5 (Section)**.** **Section** of a sheaf $\mathscr{F}$ is just an element of $\mathscr{F}(U)$, in partacular, if $s \in \mathscr{F}(X)$, we call it a **global section**.

Definition 1.6 (Stalk)**.** For $x \in X$, we define the **stalk** at x $\mathscr{F}_x$ as

$$\mathscr{F}_x := \varinjlim_{\substack{x \in U \\ U \text{ open}}} \mathscr{F}(U) = (s, U)/\sim$$

where the equivalen relationship is $(s, U) \sim (t, V)$ if there exist open $W \subseteq U \cap V$ such that $s_W = t_W$, where s_W means the image of restricting s to W.

See we have a natural morphism $\mathscr{F}(U) \to \mathscr{F}_x$ by $s \mapsto \overline{(s, U)}$, denote the image as s_x.

Remark 1.7. s_x is not determined by its valume at x, but what's the value at a point? Now, we consider it as a continuous function to get an intuition, see x and $2x$, they are both 0 at 0, but there is no neighborhood U of 0 such that they coinside. You can also see the nodal curves we mentioned at the begining.

2 Lecture 2.

22/8/31.

Fact 2.1. $\mathscr{F}_x$ is an abelian group as

$$[(s_1, U_1)] + [(s_2, U_2)] := [s_1|_{U_1 \cap U_2} + s_2|_{U_1 \cap U_2}, U_1 \cap U_2]$$

Just check it by computation, I skip the proof or, leave it as an exercise lol.

Picture 1

Remark 2.2. Take $U \in \mathcal{D}(x)$, $\mathscr{F}(U) \to \mathscr{F}_x$ is a homomorphism of abelian groups by

$$s \mapsto [(s, U)]$$

Definition 2.3 (Germ)**.** $s \in \mathscr{F}(U)$, $x \in U$, define $s_x := [(s, U)] \in \mathscr{F}_x$ which is called the **germ of s at x**.

Remark 2.4. Germ at $x \neq$ value at x which we have seen before.

- **Functuin on a set** which is a generaliztion of traditional notion of function. X = a set, $\mathscr{A} = \{A_x | x \in X\}$ a family of abelian groups indexed by X.

Define an $\mathscr{A}$-valued function as a map.

$$s : X \to \bigsqcup_{x \in X} A_x$$
$$x \mapsto s(x) \in A_x$$

See that the value area for every point of domain is different, we can also see this phenomenon in affine scheme case: $f \in A$, $f \to A/p \hookrightarrow \operatorname{Frac}(A/p)$.
Let $\mathscr{F}$ be a presheaf on a topological space X, $U \subseteq X$ open subset, take $s \in \mathscr{F}(U)$, $s : U \to \bigsqcup_{x \in U} \mathscr{F}_x$ is a $\mathscr{F}$-valued function on U.
Compare it with the valued functions:

(1) $A_x = k$ e.g. $k = \mathbb{R}, \mathbb{C}$, even finite fields.

(2) Given a continuous function $s : U \to \mathbb{R}$

$$s : U \to \bigsqcup \mathcal{O}_{U,x} \xrightarrow{\sigma} \bigsqcup_{x \in X} \mathbb{R}$$

by

$$x \mapsto s_x \mapsto s_x(x)$$

Where $\mathcal{O}_{U,x} :=$ stalk of the presheaf of continuous functions on U.
$\mathcal{O}_{U,x} \xrightarrow{\sigma_x} k \Rightarrow \ker(\sigma_x) := \{s_x \in \mathcal{O}_{U,x} | s_x(x) = 0\}$.

Fact 2.5. $\ker(\sigma_x)$ is the unique maximal ideal of $O_{U,x}$, hence

$$\mathcal{O}_{U,x} \xrightarrow{\sigma_x} \mathcal{O}_{U,x}/m_x = \mathbb{R}$$

Remark 2.6. It is the notion we considered in manifolds, we want to glue the local models to get a global one.

• Presheaf is **NOT** good as we expect ! See the following strange cases:

(1) Take nonzero $s \in \mathscr{F}(U)$ which induces $s : U \to \bigsqcup_{x \in U} \mathscr{F}_x$ by $x \mapsto s_x$. It may happen that $s_x = 0$ for all $x \in U$.

Example 2.7. Take $X = \mathbb{R}$, let

$$\mathscr{F}(U) = \begin{cases} \{f : U \times U \to \mathbb{R} \mid f \text{ continuous } \} & U \neq \varnothing \\ 0 & U = \varnothing \end{cases}$$

$\Delta := \{(x, x) \mid x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$, choose a continuous $s : U \times U \to \mathbb{R}$ such that $s = 0$ in a neighborhoodof $\Delta \cap U \times U$, but $s \neq 0 \Rightarrow s_x = 0$ but $s \neq 0$ in global!

picture 2

(2) Compatible local sections maybe can **NOT** be glued to a global section.

Example 2.8. $X = \{x, y\}$ a set of two points with discrete topology.

$A = \mathbb{Z}$.

$\mathscr{F} =$ constant presheaf on X with A.

For any open subset $U \subseteq X$, let

$$\mathscr{F}(U) = \begin{cases} A & \text{if } U \neq \varnothing \\ 0 & \text{if } U = \varnothing \end{cases}$$

Take $s_1 \in \mathscr{F}(x)$ such that $s_1 = 1$, and $s_2 \in \mathscr{F}(y)$ such that $s_2 = 0$, but we can not find a global section such that its restriction to each open subset is we want.

To deal with these strange cases, we arrive at **sheaf**.

Definition 2.9 (Sheaf)**.** Let $\mathscr{F}$ be a presheaf on X, then $\mathscr{F}$ is a sheaf if every local sections which are compatible can be **glued** to a **unique** global one.

picture 3

We have a natural way to get a sheaf from a presheaf(which is unique by universal property, for more details, see [Har77]).

Definition 2.10 (Sheafification)**.** Let $\mathscr{F}$ be a presheaf on X, the sheafification of $\mathscr{F}$ is a sheaf $\widetilde{\mathscr{F}}$ on X defined as following:

$$\widetilde{F}(U) := \{s : U \to \bigsqcup_{x \in U} \mathscr{F}_x \mid s \text{ satifies } (a) \text{ and } (b)\}$$

(a) $s_x \in \mathscr{F}_x$,

(b) $\forall x \in U$, there exists $W \in \mathcal{D}(x)$ and $t \in \mathscr{F}(W)$ such that $t(y) = s(y)\ \forall y \in W$, see it means s is defined locally! Since in every point, it coinsides with an element in a neighborhood.

picture 4

2.1 §B. Morphisms between Sheaves

After we get objects, we talk about morphisms between them.

Definition 2.11. Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves on X, a **morphism** $\varphi : \mathscr{F} \to \mathscr{G}$ is just a natural transform, **isomorphism** when it has an inverse.

Example 2.12. Let $\mathscr{F}$ be a presheaf on X, $\widetilde{\mathscr{F}}$ the sheafification of $\mathscr{F}$, then there exists a natural morphism from $\mathscr{F}$ to $\widetilde{\mathscr{F}}$

$$\mathscr{F}(U) \to \widetilde{\mathscr{F}}$$

by

$$s \mapsto (x \mapsto s_x)$$

Remark 2.13. $\varphi : \mathscr{F} \to \widetilde{\mathscr{F}}$ induces an isomorphism $\mathscr{F}_x \simeq \widetilde{\mathscr{F}_x}$ forall $x \in X$.

Definition 2.14 (Germ of morphisms)**.** Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves on X, let $x \in X$, the **germ** of φ at x is defined as

$$\begin{aligned} \varphi_x : \mathscr{F}_x &\to \mathscr{G} \\ [(s, U)] &\mapsto [(\varphi_U(s), U)] \end{aligned}$$

Definition 2.15 (Subsheaf)**.** Let $\mathscr{F}$ and $\mathscr{G}$ be presheaves on X.

(1) $\mathscr{G}$ is called a subpresheaf of $\mathscr{F}$ if $\mathscr{G}(U)$ is a subgroup of $\mathscr{F}(U)$

(2) If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, then call $\mathscr{G}$ a subsheaf of $\mathscr{F}$.

Example 2.16. $X = \mathbb{R}^n$.
$\mathscr{F}$ = sheaf of continuous functions on X.
$\mathscr{G}$ = sheaf of C^∞ functions on X.
Then $\mathscr{G}$ is a subsheaf of $\mathscr{F}$.

Definition 2.17 (Kernel)**.** Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves, define the **kernel presheaf** $\ker \varphi$ as

$$\ker(U) := \ker\colon \varphi_{\mathrm{U}}\colon \mathscr{F}(\mathrm{U}) \to \mathscr{G}(\mathrm{U})$$

Exercise 2.18. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, then $\ker(\varphi)$ is a sheaf.

Proof. We can prove this exercise in two ways:

(1) My solution is using the sheafification functor $\tilde{}$ is right adjoint, hence commute with ker, so $\ker \circ \tilde{} \simeq \tilde{} \circ \ker$.

(2) Thanks to Shengyu Hou, we can prove it by 3×3 lemma, and I will write it next time.

□

Definition 2.19 (Image presheaf)**.** Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves, the **image presheaf** $\mathrm{preim}(\varphi)$ is a subsheaf of $\mathscr{G}$ defined as

$$\mathrm{preim}\,\varphi(U) := \mathrm{im}\,\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$$

Remark 2.20. Even $\mathscr{F}$ and $\mathscr{G}$ are sheaves, $\mathrm{preim}\,\varphi$ may **NOT** be a sheaf!

Definition 2.21 (Image sheaf)**.** Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves, the **image sheaf** is the sheafification of $\mathrm{preim}(\varphi)$.

Definition 2.22. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

(1) φ is injective if $\ker(\varphi)$ is zero sheaf.

(2) φ is surjective if $\mathrm{im}(\varphi) = \mathscr{G}$(see it is after sheafification, $\mathrm{preim}(\varphi)$ may not be $\mathscr{G}$).

Remark 2.23. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

(1)

$$\begin{aligned}\varphi \text{ is injective } &\iff \forall U \in \mathcal{D}(x),\ \varphi_U \text{ is injective}\\ &\iff \forall x \in X,\ \varphi_x : \mathscr{F}_x \to \mathscr{G}_x \text{ is injective.}\end{aligned}$$

(2) φ is surjective $\iff \forall x \in X$, $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective.

(3) φ is isomorphic $\iff \forall x \in X$, $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is isomorphic.

Remark 2.24. Let $\mathscr{F}$ be a presheaf, $\mathscr{G}$ a sheaf such that $\mathscr{F}$ is a subpresheaf of $\mathscr{G}$, then

$$\mathscr{F} \subseteq \widetilde{\mathscr{F}} \subseteq \mathscr{G}$$

Definition 2.25 (Quotient and cokernel)**. I will write this next time**

2.2 §C. Base Change

Definition 2.26 (Direct image)**.** Let $f : X \to Y$ be a continuous map between topological spaces. Let $\mathscr{F}$ be a sheaf on X, then the **direct image** $f_*\mathscr{F}$ of $\mathscr{F}$ is defined as:

(1) $f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$, $U \in \mathcal{D}(Y)$.

(2) $\mathrm{res}_{U,V}^{f_*\mathscr{F}} = \mathrm{res}^{\mathscr{F}}_{f^{-1}(U),f^{-1}(V)}$.

picture

Example 2.27. Let X be a topological space, Y a single point, $f : X \to \{\mathrm{pt}\} = Y$, $\mathscr{F}$ is a sheaf on X. See $f_*\mathscr{F}(Y) = \mathscr{F}(X)$, namely $f_*\mathscr{F}$ is taking global section.

Definition 2.28 (Inverse image)**.** Let $f : X \to Y$ be a continuous map between topological spaces, $\mathscr{G}$ a sheaf on Y. Define a presheaf $Pf^{-1}\mathscr{G}$ as following:

$$Pf^{-1}\mathscr{G}(U) := \varinjlim_{\substack{f(U)\subseteq V\\ V\in\mathcal{D}(Y)}} \mathscr{G}(V) \quad \forall U \in \mathcal{D}(X),$$

i.e. $Pf^{-1}\mathscr{G}(U) = \{(s, V) \mid s \in \mathscr{G}(V), f(U) \subseteq V\}/\sim$, more explicitly, $(s, V) \sim (s', V') \iff \exists W \in \mathcal{D}(Y)$, $f(U) \subseteq W \subseteq V \cap V'$, $s\,|_W = s'\,|_W$.

Definition 2.29 (Inverse image)**.** The **inverse image** $f^{-1}\mathscr{G}$ is the sheafification of $Pf^{-1}\mathscr{G}$

Example 2.30. Let $\mathscr{F}$ be a sheaf on X, $i : U \hookrightarrow X$ the natural inclusion where U is an open subset of X. Define $\mathscr{F}|_U$ as following:

$$\mathscr{F}|_U(V) := \mathscr{F}(V), \; V \subset U \text{ both open subsets.}$$

Then $i^{-1}\mathscr{F} = \mathscr{F}|_U$.

•Compare stalks under base change.

(1) Direct image $f : X \to Y$ continuous, $x \in X$ and $y = f(x) \in Y$, $\mathscr{F}$ a sheaf on X, then f induces

$$\begin{aligned} f_x : (f_*\mathscr{F})_y &\to \mathscr{F}_x \\ [(s, U)] &\mapsto [(s, f^{-1}(U))] \end{aligned}$$

Remark 2.31. In general, f_x is neither injective nor surjective.

3 Lecture 3.

22/9/5

More on $f_x : (f_*\mathscr{F})_y \to \mathscr{F}_x$.

Example 3.1. 1) $f : \mathbb{R}^2 = X \xrightarrow{1_{\text{id}}} Y = \mathbb{R}^2$, $(x_1, x_2) \mapsto (x_1, x_2)$

X : Euclidean topology,

Y : cofinite topology(closed subsets are finite subsets),

o : the original point,

$\mathscr{F}$: sheaf of $\mathbb{R}$-valued continuous functions, sheaf on X.

Take $h : \mathbb{R}^2 \to \mathbb{R}$ continuous with $h|_{\overline{B(0,1)}} = 0$ and $h|_{X \setminus \overline{B(0,1)}}$ does not vanish. See $h_o = 0$ in $\mathscr{F}$, but not 0 in $f_*\mathscr{F}$, hence f_x is not injective.

Remark 3.2. If f is a homeomorphism, then f_x is an isomorphism.

• Stalk of inverse image.

$f : X \to Y$ continuous, $x \in X$, $y = f(x)$, $\mathscr{F}$ sheaf on Y.

$$\begin{aligned} f_x : (f^{-1}\mathscr{F})_x &\to \mathscr{F}_y \\ [(s, U)] &\mapsto [(s, V)] \end{aligned}$$

where $s \in \mathscr{F}(V), f(U) \subset V$, see $[(s, U)]$ is really a representative of elements of $(f^{-1}\mathscr{F})_x$.

Proposition 3.3. f_x is an isomorphism.

•Exactness under Base Change

Definition 3.4. $\cdots \to \mathscr{F}_i \xrightarrow{\varphi_i} \mathscr{F}_{i+1} \xrightarrow{\varphi_{i+1}} \mathscr{F}_{i+2} \to \cdots$ is exact if $\ker(\varphi_{i+1}) = \operatorname{im}(\varphi_i)$ for any i.

Proposition 3.5. (1) f_* is left exact.

(2) f^{-1} is exact, since f_x is an isomorphism.

(3) $f^{-1} \dashv f_*$

Remark 3.6. All concepts of sheaf of abelian groups can be defined for sheaf of rings(commutative with identity).

3.1 §D. Ringed Space

Definition 3.7.

(1) A ringed space is a pair $(X, \mathcal{O}_X)$: X a topological space, and $\mathcal{O}_X$ a sheaf of rings on X.

(2) $\mathcal{O}_X$ is called structure sheaf.

(3) Elements of $\mathcal{O}_X(U)$ are called regular functions on U, where U is an open subset of X.

Example 3.8. (1) $(X, \mathcal{O}_X) = \begin{cases} X \text{ topological space} \\ \mathcal{O}_X = \text{sheaf of } \mathbb{R}\text{-valued continuous function} \end{cases}$

(2) $(X, \mathcal{O}_X) = \begin{cases} X \text{ differential manifold} \\ \mathcal{O}_X = \text{sheaf of } \mathbb{C}^\infty \text{ functions} \end{cases}$

(3) $(X, \mathcal{O}_X) = \begin{cases} X \text{ complex manifold} \\ \mathcal{O}_X = \text{sheaf of holomorphic functions} \end{cases}$

Definition 3.9 (Locally ringed space)**.** $\mathcal{O}_{X,x}$ is a local ring, $\forall x \in X$.

Definition 3.10.

(1) Morphism between ringed space is a pair $(f, f^\#)$

$$f : X \to Y \text{continuous}$$
$$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X \text{ a morphism of sheaf of rings(hence natural)}$$

(2) For locally ringed soace, we need

$$f_x^{\#} : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \text{ is a local morphism}$$
$$[(s,U)] \mapsto [(f^{\#}(s), f^{-1}(U))]$$

(3) Isomorphism:

(a) f is a homeomorphism.

(b) $f^{\#}$ is an isomorphism of sheaves of rings.

Example 3.11 (Local model of diff/cplx mfd)**.** $(X, \mathcal{O}_X)$ = differential/complex manifold then $\forall x \in X$, $\exists U \subset X$ such that

$$(U, \mathcal{O}_X\mid_U) \simeq (B_{(0,1)}, \mathcal{O}_{B(0,1)})$$

4 Lecture 4.

22/9/7.

We still talk about f_x.

Example 4.1. $\widehat{S}$ =Riemann Sphere, $\widehat{S} = \mathbb{C} \cup \{\infty\}$ compact complex manifold.

$$\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$$
$$z \mapsto \frac{1}{z}$$

Take $X = \widehat{S} \sqcup \widehat{S}$ with Euclidean topology.

$\mathscr{F}$ = sheaf of holomorphic functions on X.

$Y = \widehat{S} \sqcup \widehat{S}$ with trivial topology(open sets are only $\varnothing$ and Y).

$f : X \xrightarrow{\text{id}} Y$.

$(f_*\mathscr{F})_x = \mathscr{F}(X) = \{g : X \to \mathbb{C} \mid g\mid_{\widehat{S}_i} = \text{constant}\}$ so we may have $g_1 = 0$ and $g_2 = 1$.

Consider $f_x : \mathscr{F}(X) = (f_*\mathscr{F})_x \to \mathscr{F}_x$.

(i) Not injective. $\begin{cases} g = 0 & x \in \widehat{S} \\ g = 1 & x \notin \widehat{S} \end{cases} \Rightarrow g \neq 0$ but $g_x = 0$.

(ii) $s \in \mathscr{F}_x$ such that $'(s,U)'$ is not a constant(in any neighborhood of x) $\Rightarrow$ there does not exist $g \in \mathscr{F}(X)$ such that $g_x = s$.

II. Affine Algebraic Sets

Affine algebraic sets= topological space of local model in algebraic geometry.

k = a field, e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{F}_p$...

4.1 §A. Affine Algebraic Sets

Some notations:

$$\begin{aligned} n \in \mathbb{Z}_{\geq 0} \quad \mathbb{A}_k^n &= \text{affine } k\text{-space of dimension } n \\ &= \{(x_1, ..., x_n \mid x_i \in k)\} \\ &= k^n (\text{as sets}) \end{aligned}$$

$k[x_1, ..., x_n]$ =ring of polynomials in n variables.

Definition 4.2. Let $S \subset k[x_1, ..., x_n]$, we define

$$Z(S) := \{x \in \mathbb{A}_k^n \mid \forall F \in S, \quad F(x) = 0\}$$

We call $Z(S)$ the affine algebraic set defined by S(it is namely the common zero locus of elements of S).

If $S = \{F_1, ..., F_n\}$ is a finite set, we denote $Z(S)$ by $Z(F_1, ..., F_n)$.

Example 4.3. 1) $Z(1) = \varnothing$, $Z(0) = \mathbb{A}_k^n$.

2) If $n = 1$ and $S \neq \varnothing$, then $Z(S)$ is a finite set. Conversely, given a finite set $\{x_1, ..., x_m\} \subset \mathbb{A}_k^1$, just take $F = \prod_{i=1}^m (X - x_i)$, hence $Z(F) = \{x_1, ..., x_m\}$. So in $\mathbb{A}_k^1$ case, affine algebraic set= finite set, just cofinite topology.

4.2 §B. Zariski Topology

Lemma 4.4. $S \subset k[x_1, ..., x_n]$, $I =$ ideal generated by S, then $Z(I) = Z(S)$.

Corollary 4.5. It is enough to consider the affine algebraic sets defined by ideals.

(i) $\cap_i Z(I_i) = Z\left(\sum I_i\right)$ arbitray intersection.

(ii) $\cup^m Z(I_i) = Z(\cap^m I_i)$. finite union.

Definition 4.6 (Zariski topology)**.** Closed subsets of $\mathbb{A}_k^n$ in Zariski topology is the affine algebraic sets of $\mathbb{A}_k^n$.

For $V \subset \mathbb{A}_k^n$, it can inherit Zariski topology from $\mathbb{A}_k^n$.

Proposition 4.7. $F : \mathbb{A}_k^n \to k = \mathbb{A}_k^1$ where F is a polynomial, F is continuous in Zariski topology.

Proof. Let $V = \{a_1, ..., a_n\} \subset \mathbb{A}_k^1$ closed subset(=finite set), $F^{-1}(V) \cup F^{-1}(a_i) = V\left(\prod (F - a_i)\right)$ closed in $\mathbb{A}_k^n \Rightarrow F$ is continuous. □

Remark 4.8. Zariski topology the weakest topology on $\mathbb{A}_k^n$ such that $F : \mathbb{A}_k^n \to k$ is continuous, where k with cofinite topology.

Remark 4.9. Zariski topology is very different from the Euclidean topology.

Let's talk some characters of Zariski topology:

(a) Closed subsets are 'very small'(measure=0), opens are 'very big'! E.g. $\mathbb{A}_k^1$ Zariski=cofinite $\not\Rightarrow$ open ball in Euclidean.

(b) Zariski topology is <u>NOT</u> Hausdorff. E.g. $k =$ infinite field, $U_1, U_2 \subset \mathbb{A}_k^1$ then $U_1 \cap U_2$ never empty.

(c) $\mathbb{A}_k^n \overset{\text{set}}{=} \underbrace{k \times \cdots k}_{\text{n-times}} \overset{\text{set}}{=} \underbrace{\mathbb{A}_k^1 \times \cdots \mathbb{A}_k^1}_{\text{n-times}}$. However, Zariski on $\mathbb{A}_k^n \neq$ product topology on $\mathbb{A}_k^1 \times \cdots \mathbb{A}_k^1$.

(d) $\forall F \in k[x_1, ..., x_n]$, define $D(F) := \mathbb{A}_k^n \setminus Z(F)$ is open, then $\{D(F) \mid F \in k[x_1, ..., x_n]\}$ form a base for Zariski topology.

(e) Many difficulities in algebraic geometry to study local properties at a point are caused by (a) and (b).

4.3 §C. Ideals of Affine Algebraic Sets

Definition 4.10. Let $V \subset \mathbb{A}_k^n$ be a subset, then define $I(V) := \{F \in k[x_1, ..., x_n \mid F(v) = 0 \quad \forall v \in V]\}$, it is obviously an ideal.

Fact 4.11.

(1) If V is an affine algebraic set, then $Z(I(V)) = V$. It is obvious that $V \subset Z(I(V))$, conversely, $V = Z(J)$ where J is an ideal, then $J \subset I(V)$, $V = Z(J) \supseteq Z(I(V))$, hence equal.

(2)

$$\{\text{affine algebraic set in } \mathbb{A}_k^n\} \to \{\text{ideals in } k[x_1, ..., x_n]\}$$
$$V \mapsto I(V)$$

is injective, hence if $I(V_1) = I(V_2)$ then $V_1 = V_2$. In particular, if $W \neq V$, then $\exists f$ polynomial that $f \mid_W = 0, f \mid_V \neq 0$, a polynomial can distinguish two affine algebraic sets.

(3) $J \subset Z(I(J))$, in general not equal unless J is radical, e.g. $J = (x^2) \Rightarrow V(J) = \{0\} \Rightarrow I(\{0\}) = (x) \neq (x^2)$ but $\sqrt{(x^2)}$.

(4) If k is <u>infinite</u>, then $I(\mathbb{A}_k^n) = 0$, e.g. if $k = \mathbb{F}_p$, consider $F = \prod_{i=1}^p (X - a_i)$, then $F \in I(\mathbb{A}_k^1)$.

Proof. n=1. Since every $F \in k[X]$ has only finite roots, then for any subset S of $k[X]$, $Z(S) \neq \mathbb{A}_k^1$ since $\mathbb{A}_n^1$ has infinite elements.

n=2. Take $P \in k[X_1, ..., X_n]$ such that $P \neq$ constant, then we can write $P = a_r(X_1, ..., X_{n-1})X_n^r + \cdots$, assume $r \geq 1$ and $a_r(X_1, ..., X_{n-1}) \neq 0$.

$\Rightarrow$ By induction, $\exists (b_1, ..., b_{n-1}) \in \mathbb{A}_k^{n-1}$ such that $a_r(b_1, ..., b_{n-1}) \neq 0$

$\Rightarrow f(X_n) = P(b_1, ..., b_{n-1}, X_n) = a_r(b_1, ..., b_{n-1})X_n^r$ which has only finitely mant solutions in k.

$\Rightarrow$ there exists $b_n \in k$ such that $f(b_n) \neq 0$, hence $P \notin I(\mathbb{A}_k^n)$.

$\Rightarrow I(\mathbb{A}_k^n) = \varnothing$.

n=3. $a = (a_1, ..., a_n) \in \mathbb{A}_k^n$, then $I(\{a\}) = (x_1 - a_1, ..., x_n - a_n)$

□

Theorem 4.12 (Hilbert Basis Thm)**.** *Let A be a Noetherian ring, then the polynomial ring $A[X]$ is Noetherian, more generally, so does $A[[X]]$.*

Corollary 4.13. Given an affine algebraic set $V \subset \mathbb{A}_k^n$, then $\exists f_1, ..., f_r \in k[x_1, ..., x_n]$ such that $V = Z(f_1, ..., f_r)$, i.e. V is the intersection of finitely many hypersurfaces!

4.4 §D. Hilbert Nullstellensatz

$$\{\text{affine algebraic sets}\} \rightleftarrows \{\text{ideals of } k[x_1, ..., x_n]\}$$
$$V \mapsto I(V)$$
$$Z(J) \leftarrow J$$
$$V = Z(I(V))$$
$$J \subset I(V(J))$$

Here is a proposition from [AM94] .

Proposition 4.14 ([AM94].prop.7.9)**.** $k =$ a field, $R =$ finitely generated k-algebra.
If R is a field, then R is a finite extension of k. In particular, if k is algebraically closed, then $R = k$.

Theorem 4.15 (Hilbert Nullstellensatz)**.** *$k =$ algebraic closed field (e.g. $\mathbb{C}, \overline{\mathbb{Q}}$).*
$J =$ an ideal of $k[x_1, ..., x_n]$, then $I(Z(J)) = \sqrt{J}$.

Example 4.16. $n = 1$, $k = \mathbb{R}$, $J = (x^2 + 1) \subset \mathbb{R}[x]$, then $Z(J) = \varnothing$, but $1 \notin \sqrt{(x^2 + 1)}$.

Corollary 4.17. If k is algebracially closed, then there is a one to one correspindence between affine algebraic sets in $\mathbb{A}_k^n$ and radical ideals of $k[x_1, ..., x_n]$

$$\{\text{affine algebraic sets in } \mathbb{A}_k^n\} \longleftrightarrow \{\text{radical ideals in } k[x_1, ..., x_n]\}$$
$$V \mapsto I(V)$$
$$Z(J) \leftarrow J$$

See $V \mapsto I(V) \mapsto Z(I(V)) = V$, see $I(V)$ is a radical by Hilbert(if f vanishes on V, then $f^r \in I(V)$ for some r) and $J \mapsto Z(J) \mapsto I(Z(J)) = \sqrt{J} = J$, hence bijection.

Corollary 4.18 (Weak form)**.** If k is algebraically closed, then I contains $1 \iff \{f_i\}$ the generators of I have no common zeros.

5 Lecture 5.

22/9/14.

•Recall:

(1) Affine algebraic set.

(2) Zariski topology.

(3) Zariski topology on V.

(4) Hilbert Nullstellensatz.

III. Affine Algebraic Variety

An affine algebra variety is a locally ringed space $(V, \mathcal{O}_V)$ where Let V be an affine algebraic setwith Zariski topology on it, $\mathcal{O}_V$ = sheaf of regular functions.

In this chapter, we aim to define the regular functions on open subsets of V.

1. In differential geometry, we consider C^∞ functions.

2. In complex geometry, we consider holomorphic functions.

3. In algebraic geometry, we consider functions defined by polynomials.

5.1 §A. Regular Functions

Let $V \subseteq \mathbb{A}_k^n$ be an affine algebraic set.

Definition 5.1 (Coordinate ring of V)**.**

$$\begin{aligned} A(V) :&= \{f : V \to k \mid \exists F \in k[x_1, ..., x_n] \text{ such that } F \mid_V = f\} \\ &= \{F \in k[x_1, ..., x_n]/ \sim\} \end{aligned}$$

Where $F \sim F' \iff F \mid_V = F' \mid_V \iff F - F' \in I(V)$, hence $A(V) = k[x_1, ..., x_n]/I(V)$. $A(V)$ is called the **coordinate ring** of V.

• Basic facts of Zariski topology on V.

(1) Given $f \in A(V)$, $D(f) = \{x \in V \mid f(x) \neq 0\}$ is open in V(just take f to F, $D(F) \cap V$).

(2) Given an open $U \subseteq V$, then $\exists f_1, ...f_r \in A(V)$ such that $U = \cup_{i=1}^n D(f_i)$. In particular, $\{D(f_i)\}$ $f \in A(V)$ form a base of the Zariski topology on V. Similiar proof like above one.

Remark 5.2. We call U **quasi-affine**.

•Assume we have defined $\mathcal{O}_V$, then we want:

(1) Global: $A(V) \subseteq \mathcal{O}_V(V)$.

(2) Local: on $D(f)$, where $f \in A(V)$, $\mathcal{O}_V(D(f)) = \left\{\frac{g}{f^n} : D(f) \to k \mid n \in \mathbb{Z}_{\geq 0}, g \in A(V)\right\}$, see its elements are k-valued functions.

Now, we come to the sheaf of regular functions.

Definition 5.3 (1st. Regular functions on subsets)**.** Let $U \subseteq V$ be an open subset,

$$\mathcal{O}_V(U) := \left\{ s : U \to k \middle| \text{such that } \exists U = \cup_{i=1}^r D(f_i), f_i \in A(V), s \mid_{D(f_i)} = \frac{g_i}{f_i^{n_i}} \in \mathcal{O}_V(D(f_i)) \right\}$$

where $g_i \in A(V)$ and $n_i \in \mathbb{Z}_{\geq 0}$, and by Hilbert basis theorem, we can find a finite open cover of U.

Example 5.4.

(1) $V = \{x_1x_2 - x_3x_4 = 0\} \subseteq \mathbb{A}_k^4$

(2) $k = \mathbb{R}$, $V = \mathbb{A}_k^1$, $g = x^2 + 1 \Rightarrow D(f) = V \Rightarrow \frac{1}{f} \in \Gamma(V, \mathcal{O}_V) \Rightarrow A(V) \subsetneq \Gamma(V, \mathcal{O}_V)$(since $\mathbb{R}$ is not algebraic closed).

Before we introduce the 2nd definiton of sheaf of regular functions, let's recall the **sheafification of a presheaf**.

1) First, we define $\mathscr{F}_x$ =regular functions in a small neighborhoodof x.

2) Second, we define $\mathscr{F}(V)$ =gluing compatible local functions of $\mathscr{F}_x$ $\forall x \in U$.

Definition 5.5 (Regular functions at a point)**.** $\forall x \in V \subseteq \mathbb{A}_k^n$, we define

$$\mathcal{O}_{V,x} = \left\{ \frac{g}{f} \middle| f, g \in A(V), f(x) \neq 0 \right\} / \sim$$

Where $\frac{g}{f} \sim \frac{g'}{f'}$ if $\exists f'' \in A(V)$, $x \in D(f'') \iff f''(x) \neq 0$, such that

$$\frac{g}{f}\bigg|_{D(f'')} = \frac{g'}{f'}\bigg|_{D(f'')}$$

Which is equivalent to $f''(gf' - fg') = 0$ in $A(V)$(see we can shrink $D(f'')$ small enough to be contained in $D(f)$, taking intersection is enough)

Proposition 5.6. there exists an isomorphism of rings $\mathcal{O}_{V,x} \simeq A(V)_{\mathfrak{p}_x}$, $[\frac{g}{f}] \mapsto [\frac{g}{f}]$, where $\mathfrak{p}_x = \{f \in A(V) \mid f(x) = 0\}$.

Definition 5.7 (2nd. Regular functions on general open subsets)**.** $U \subseteq V$.

$$\mathcal{O}_V(U) = \left\{ s : U \to \coprod_{x \in X} \mathcal{O}_{V,x} \middle| s|_{D(f)} = \frac{g}{f^n} \quad \forall x \in U, \text{for some } f, g \in A(V),\ x \in D(f) \subseteq U, n \in \mathbb{Z}_{\geq 0} \right\}$$

Proposition 5.8 (Proterties of $\mathcal{O}_V$)**.**

(1) $\mathcal{O}_V$ is a sheaf of rings on V.

(2) $\mathcal{O}_{V,x} \simeq A(V)_{\mathfrak{p}_x}$.

(3) If $k = \overline{k}$, then $\mathcal{O}_V(D(f)) = A(V)_f$, it is exactly the local picture we want!

Remark 5.9. $k = \mathbb{R}$,

$$\begin{aligned} s = \frac{1}{x^2+1} &\in \mathcal{O}_{\mathbb{A}_k^1}(\mathbb{A}_k^1) \\ &\notin A(\mathbb{A}_k^1) = k[x]/(0) = k[x], \text{ recall } I(\mathbb{A}_k^1) = (0) \end{aligned}$$

Remark 5.10. (3) needs a long and maybe tedious proof you can find it in [Har77] , but it is worth a try, since it is a basic trick in algebraic geometry.

5.2 §B. Affine Algebraic Variety

Definition 5.11. A locally ringed space $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is the sheaf of k-valued functions, is called an **affine algebraic varieties** if there exists an affine algebraic set$V$ and a homeomorphism $f : X \to V$ such that $(f, f^\#) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ is an isomorphism of locally ringed spaces, where

$$\begin{aligned} f_U^\# : \mathcal{O}_V(U) &\to f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U)) \quad U \subseteq V \text{ open} \\ s &\mapsto s \circ f \end{aligned}$$

Remark 5.12.

(1) $\mathrm{Hom}_{Var}(X, Y)$ = morphisms of locally ringed space where $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ are affine algebraic varieties.

(2) The definition of affine algebraic variety is intrinsic, while that of an affine algebraic set is <u>NOT</u>(depends on the embedding into $\mathbb{A}_k^n$).

Proposition 5.13. Let $V \subset \mathbb{A}_k^n$ be an affine algebraic set, $f \in A(V)$, see $(D(f), \mathcal{O}_V \mid_{D(f)})$ is also an affine algebraic variety, since $\mathcal{O}_V(D(f)) = i_* \mathcal{O}_V$, where $i : D(f) \hookrightarrow V$.

Example 5.14. $V = \mathbb{A}_k^1$, $f = X$, $D(X) = \mathbb{A}_k^1 \setminus \{(0)\}$, $V' = \{XY = 1\}$, $p(x, y) = x \Rightarrow p(V') = D(X)$(bijection).
Define $\Phi : \mathbb{A}_k^n \setminus V(F) \to \mathbb{A}_k^{n+1}$ by $(x_1, ..., x_n) \mapsto \left(x_1, ..., x_n, \frac{1}{f(x_1, ..., x_n)}\right)$.
Define $\Phi^{-1} : \mathbb{A}_k^{n+1} \to \mathbb{A}_k^n$ by $(x_1, ..., x_n, x_{n+1}) \mapsto (x_1, ..., x_n)$, $X = \Phi(V)$ is an affine algebraic setin $\mathbb{A}_k^{n+1}(V(F(x_1, ..., x_n)x_{n+1} - 1))$.

Exercise 5.15. $\Phi : D(f) = V \cap D(F) \to X$ is an isomorphism, pullback of a regular function is still a regular function.

Corollary 5.16. Any open subset of an affine algebraic varietyis covered by open subsets which are affine algebraic varieties.

Remark 5.17. Affine algebraic variety is the local model in algebraic geometry.

Proposition 5.18. Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be affine algebraic varieties, then there exists one-to-one correspondence.

$$\mathrm{Hom}_{Var}(X, Y) \overset{1:1}{\longleftrightarrow} \mathrm{Hom}_{k\text{-}alg}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

Corollary 5.19. If $k = \overline{k}$. Cat{affine algebraic variety} $\simeq$ Cat{reduced finitely generated k-algebra}.

Example 5.20 (Bijection $\neq$ isomorphism)**.** $f : \mathbb{A}_{\mathbb{R}}^1 \to \mathbb{A}_{\mathbb{R}}^1$ by $x \mapsto x^3$ is an bijection.
$f_o^{\#} : k[x]_x \mathcal{O}_{\mathbb{A}_k^1, o} \to \mathcal{O}_{\mathbb{A}_k^1, o} = k[x]_x$ is $x \mapsto x^3$ is not an isomorphism of local rings. Hence f is not an isomorphism.

6 Lecture 6.

22/9/19.
●: compare two definitions of affine algebraic varieties.

Definition 6.1 (1st given in the course)**.** A locally ringed space $(X, \mathcal{O}_X)$ with $\mathcal{O}_X$ a sheaf of k-valued functions($\mathcal{O}_X(U) \subseteq \{s : U \to k\}$ $U \subseteq X$ open) is called an affine algebraic variety if $\exists (V, \mathcal{O}_V)$ with V an affine algebraic set, $\mathcal{O}_V$ the sheaf of regular functions on V, $\exists f : X \to V$ such that

(a) f is a homeomorphism.

(b) $\forall\, U \subset V$ open, $f_U^\# : \mathcal{O}_V(U) \to \mathcal{O}_X(f^{-1}(U))$ with $s \mapsto s \circ f$, so k-valued function on $f^{-1}(U)$, $(f, f^\#) : (X, \mathcal{O}_X) \to (V, \mathcal{O}_V)$ is an isomorphism of locally ringed space.

Remark 6.2. See that $s \circ f$ may not in $\mathcal{O}_X(f^{-1}(U))$, but we require it belongs to it.

Definition 6.3 (2nd)**.** A locally ringed space $(X, \mathcal{O}_X)$ is called an affine algebraic variety, if there exists $(V, \mathcal{O}_V)$ with V an affine algebraic set, $\mathcal{O}_V$ sheaf of regular functions, such that there exists $(f, f^\#) : (X, \mathcal{O}_X) \simeq (Y, \mathcal{O}_V)$ an isomorphism of locally ringed space.

Question 6.4. Where is the difference?

(1) In Def 1st, we require $\mathcal{O}_X$ is a sheaf of k-valued functions, but $f^\#$ is <u>induced</u> by f, not a **priori** given in definition.

(2) In Def 2nd, a priori $\mathcal{O}_X$ may not be a sheaf of k-valued functions and $f^\#$ is a priori given in the definition.

However, we will show that they are equivalent! Def 1.$\Longleftrightarrow$Def 2.

Recall 6.5 (Locally ringed space)**.**

$$\begin{aligned}(X, \mathcal{O}_X) &= \text{a locally ringed space} \\ &= X \text{ topological space } + \mathcal{O}_X \text{ sheaf of rings } + \mathcal{O}_{X,x} \text{ local ring}\end{aligned}$$

Where $\mathcal{O}_{X,x}/m_x$ is a field, called the residue field, denoted $\kappa(x)$.
$\forall\ V \subseteq X$ open, $\mathcal{O}_X(U) \subseteq \left\{s : U \to \coprod_{x \in U} \mathcal{O}_{X,x} \mid \forall x \in U, s(x) \in \mathcal{O}_{X,x}\right\}$, see we have a natural $\varphi_U : \coprod_{x \in U} \to \coprod_{x \in U} \kappa(x)$ which induces:

$$\left\{\widetilde{s} : U \to \coprod_{x \in U} \mathcal{O}_{X,x} \to \coprod_{x \in U} \kappa(x) \mid \forall x \in U, \widetilde{s}(x) \in \kappa(x)\right\}$$

Remark 6.6. (1) $\kappa(x)$ is independent of x.

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{X,x} = \kappa(x)/m_x \\ {\scriptstyle f_x^\#}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle \widetilde{f_x^\#}} \\ \mathcal{O}_{V,f(x)} & \longrightarrow & \mathcal{O}_{V,f(x)}/m_{f(x)} = \kappa(f(x)) \simeq k \end{array}$$

The right vertical isomorphism is induced by $f_x^\#$ is a local homomorphism.

(2) φ_U is injective, $\forall\, U \subseteq X$ open:

$$\begin{array}{cccc} s & \mathcal{O}_X(U) & \xrightarrow{\varphi(U)} & \{U \to \coprod_{x\in U} \kappa(x)\} \\ \uparrow & {\scriptstyle f_U^\#}\uparrow\simeq & & \uparrow \\ \widetilde{s} & \mathcal{O}_V(f(U)) & \xrightarrow{\varphi_{f(U)}} & \{f(U) \to \coprod_{y\in f(U)} \kappa(y)\} \end{array}$$

If $s \in \ker\varphi$, then there exists $\widetilde{s} \in \mathcal{O}_V(f(U))$ such that $f_U^\#(\widetilde{s}) = s$, hence $\varphi_{f(U)}(\widetilde{s}) = 0$(see the right vertical arrow is natural), by axiom of sheaf, $\widetilde{s} = 0$, hence $s = 0$.

Hence, if $(X, \mathcal{O}_X)$ is an affine algebraic varietyin the sense of Def 2. then $\mathcal{O}_X$ is naturally a sheaf of k-valued functions on V, and $(X, \mathcal{O}_X)$ is an affine algebraic varietyin the sense of Def 1.

'affine algebraic variety' in Algebraic Geometry $\simeq$ 'open ball' in differential/complex geometry. Open subsets of affine algebraic variety= union of standard open subsets $D(f)$, $f \in \Gamma(X, \mathcal{O}_X)$, and $(D(f), \mathcal{O}_X\mid_{D(f)}$ is an affine algebraic variety.

IV. Basic Properties of Affine Algebraic Varieties

6.1 §A. Irreducibility

Example 6.7. $V = \{XY = 0\} \subseteq \mathbb{A}_k^2$, we have X-axis($Y = 0$) and Y-axis($X = 0$), hence $V = V(X) \cup V(Y)$, we decompose the variety into two varieties!

Definition 6.8 (Irreducibility)**.** Let X be a topological space, then the followings are equivalent:

(a) If $X = F \cup G$, F, G are closed subsets of X then either $X = F$ or $X = G$.

(b) If $U \cap V = \varnothing$ in X, where U, V are open then either $U = \varnothing$ or $V = \varnothing$.

(c) If $U \subseteq X$ is a nonempty open subset of X, then $\overline{U} = X$(open is dense).

Example 6.9.

(1) $\mathbb{R}$ with Euclidean topology is reducible(minus a open ball then cups the closure of the open ball).

(2) If X is Hausdorff, then only singleton space is irreducible, since for every pair of points, we can give them disjoint open neighborhoods, hence opens are not dense, and we will see later the dimension of Hausdorff space is 0 due to this property.

Proposition 6.10.

(1) $V \subseteq \mathbb{A}_k^n$ an affine algebraic set, then

$$\begin{aligned} V \text{ is irreducible } &\iff I(V) \text{ is prime} \\ &\iff A(V) \text{ is an integral domain} \end{aligned}$$

(2) V is an affine algebraic variety then V is irreducible $\iff \Gamma(V, \mathcal{O}_V)$ is an integral domian. Recall that if $\overline{k} = k$, then $\Gamma(V, \mathcal{O}_V) = A(V)$, if not, then they are different(we may have denominator is a polynomial).

Corollary 6.11. If k is infinite, then $\mathbb{A}_k^n$ is irreducible since $I(\mathbb{A}_k^n) = (0)$.

Remark 6.12. If k is finite, $\mathbb{A}_k^n$ = union of finite points, and actually every point is closed, hence not irreducible.

Question 6.13. So, can we decompose X into union of irreducible components?

Definition 6.14 (Noetherian Space)**.** A topological space X is called **Noetherian** if it satisfies *ACC* for open subsets which is equivalent to *DCC* on closed subsets.

Example 6.15. By Hilbert basis theorem, an affine algebraic set is Noetherian.

Theorem 6.16. *If X is Noetherian, then X has a unique decomposition into union of irreducible components, $X = \cup_i^n U_i$ with U_i irreducible and $U_i \subsetneq U_j$ for any i, j.*

7 Lecture 7.

22/9/23.

•From now on, we will always assume $k = \overline{k}$, $\operatorname{char} k = 0$. e.g. $\mathbb{C}$(uncountable), $\overline{\mathbb{Q}}$(countable).

7.1 §B. Dimension

•Comparing with differential/complex geometry, then main difficulty to define the dimension of an affine algebraic variety is that in general, it is NOT an open subset of an affine space $\mathbb{A}_k^n$(see in differential/complex geometry just use the dimension of local open to define dimension).

We will give three methods to define 'dimension' which are all interesting.

(1) Topological dimension

Definition 7.1. Let X be a topological space. We define the dimension of X, denoted by $\dim(X)$, to be the supremum of all integers n such that there exists a chain of distinct irreducible closed subsets

$$Y_0 \subsetneq \cdots \subsetneq Y_n$$

Remark 7.2. (1) We can think $\dim(X)$ as a definition "from small to large", like

$$\text{point} \subsetneq \text{curve} \subsetneq \text{surface} \subsetneq \cdots \subsetneq X$$

Like

picture

(2) This definition is very different from given in differential/complex geometry. Actually if X is Hausdorff, then the only irreducible component is singleton, see a Hausdorff space X is irreducible if and only if X is a singleton.

(2) Krull dimension

Recall 7.3 (Krull dimension of a ring)**.** Let A be a ring, $\mathfrak{p} \in \operatorname{Spec} A$, then we define the **height** of $\mathfrak{p}$ denoted by $\operatorname{ht}\mathfrak{p}$.

$$\operatorname{ht}\mathfrak{p} := \sup\left\{n \in \mathbb{Z}_{n\geq 0}\middle| \ \exists \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p} \quad \mathfrak{p}_i \text{ prime ideals}\right\}$$

Then the Krull dimension of A is defined as $\dim_K(A) = \sup_{\mathfrak{p}\in\operatorname{Spec} A} \operatorname{ht}(\mathfrak{p})$.

Corollary 7.4. Let V be an affine algebraic variety. $A = \Gamma(V, \mathcal{O}_V) = A(V)$, $V \subset \mathbb{A}_k^n$, $A(V) = k[x_1, ..., x_n]/I(V)$

$$\{\text{prime ideals in } A(V)\} \overset{1:1}{\longleftrightarrow} \{\text{irreducible closed subsets in } V\}$$
$$\mathfrak{p} \mapsto V(\mathfrak{p})$$
$$U \mapsto I(U)$$

Hence

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p} \mapsto Y_0 \subsetneq \cdots \subsetneq Y_n$$

Where $\mathfrak{p}_i \longleftrightarrow Y_{n-i}$, is one to one reverse ordering.

Proposition 7.5. Let V be an affine algebraic variety. Then $\dim V = \dim_K \Gamma(V, \mathcal{O}_V)$, where $\dim V$ is topological dimension, $\dim_K \Gamma(V, \mathcal{O}_V)$ is Krull dimension.

(3) Transcendence degree of field of rational functions

Fact 7.6. An affine algebraic variety is irreducible $\Longleftrightarrow \Gamma(V, \mathcal{O}_V)$ is an integral domain.

Example 7.7. $V = (xy)$ is not irreducible by $A(V) = k[x, y]/(xy)$.

Definition 7.8 (Field of rational functions)**.** Let V be an irreducible affine algebraic variety. Then the **field of rational functions** $K(V)$ of V is $\operatorname{Frac}(A(V))$.

Example 7.9. $V = (x_1x_2 - x_3x_4) \subseteq \mathbb{A}_k^n$, hence $\frac{x_1}{x_3} = \frac{x_4}{x_2}$ in $K(V)$.

Theorem 7.10 ([AM94] Chapter 11)**.** *If $k = \overline{k}$, A a domain, finitely generated k-algebra. Then* $\dim_K(A) = \operatorname{tr.deg}_k(\operatorname{Frac}(A))$.

Corollary 7.11. Let V be an irreducible affine algebraic variety. Then $\dim(V) = \operatorname{tr.deg}_k(K(V))$

Example 7.12. $\dim(\mathbb{A}^n_k) = \operatorname{tr.deg}_k(k(x_1, ..., x_n)) = n.$, which coincides with our intuition.

Theorem 7.13 (Noether's normalization lemma)**.** *Let R be a domain, finitely generated k-algebra. If $n = \operatorname{tr.deg}_k(R)$, then there exists $x_1, ..., x_n \in R$ algebraically independent over k such that R is integrally dependent over the subring $k[x_1, ..., x_n] \subseteq R$.*

Corollary 7.14. Let V be an irreducible affine algebraic variety of dimension n, then there exists a **dominant** morphism $\varphi : V \to \mathbb{A}^n_k$ which means $\overline{\varphi(V)} = \mathbb{A}^n_k$.

Proof. Since $\dim V = n$ we have $\operatorname{tr.deg}_k \Gamma(V, \mathcal{O}_V) = n$, hence $k[x_1, ..., x_n] \hookrightarrow \Gamma(V, \mathcal{O}_V)$ which induces $\varphi : V \to \mathbb{A}^n_k$, see the lemma below. □

Lemma 7.15. If $\Gamma(U, \mathcal{O}_U) \hookrightarrow \Gamma(V, \mathcal{O}_V)$, then it induces $\varphi : V \to U$ which is dominant.

4) Local dimension

Definition 7.16. Let V be an affine algebraic variety, $p \in V$ be a point. Then the local dimension $\dim_p V$ of V at p is defined by the Krull dimension of its local ring, $\dim_K \mathcal{O}_{V,p} = \dim_K A(V)_{\mathfrak{m}_p}$, where $\mathfrak{m}_p = \{f \in \Gamma(V, \mathcal{O}_V) \mid f(p) = 0\}$.

Proposition 7.17. $\dim_p V = \sup \left\{ n \in \mathbb{Z}_{n\geq 0} \middle| \exists \{p\} \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq V \quad Y_i \text{ irreducible closed} \right\}$(see singleton is closed.).

Fact 7.18. $k = \overline{k}$, let V be affine algebraic variety. Then there exists 1:1 correspondence:

$$\{\text{points in } V\} \longleftrightarrow \{\text{maximal ideals in } \Gamma(V, \mathcal{O}_V)\}$$
$$p \mapsto \mathfrak{m}_p$$

See $\mathbb{C}$ case, maximal ideals correpond to points.
It is easy to see that prime ideals in $\mathcal{O}_{V,p} \longleftrightarrow$ prime ideals contained in m_p, see

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq \mathcal{O}_{V,p} \leftrightsquigarrow \mathfrak{p}'_0 \subsetneq \cdots \subsetneq \mathfrak{p}'_n \subseteq \mathfrak{m}_\mathfrak{p} \subseteq \Gamma(V, \mathcal{O}_V)$$

Where are both chain of prime ideals.

Proposition 7.19. $\dim_K \mathcal{O}_{V,p} = \operatorname{ht}(\mathfrak{m}_p)$.

Proof.

$$\begin{aligned}
\dim_p V = \dim_K \mathcal{O}_{V,p} &= \operatorname{tr.deg}_k(\operatorname{Frac}(\mathcal{O}_{V,p})) \\
&\overset{V \text{ irre}}{=} \operatorname{tr.deg}_k(K(V)) \\
&= \dim V
\end{aligned}$$

□

Remark 7.20. If V is NOT irreducible, then the general $\dim_p V \neq \dim V$. For example, Let V be line $\cup$ pt $\subseteq \mathbb{A}_k^2$, then $\dim(V) = 1$ i.e. $\{p'\} \subsetneq l$, BUT $\dim_p V = 0$

•Depth: from large to small

Recall Top, Krull from small to large, like pt $\subsetneq$ curve $\subsetneq$ surface $\subsetneq \cdots$

Theorem 7.21 (Krull's Hauptidealsatz). *If A is Noetherian ring, $f \in A$ neither zero divisor nor unit, then $\dim_K A/(f) = \dim_K(A) - 1$ which is equivalent to every minimal prime ideal containing (f) has height 1.*

Example 7.22. Let V be an affine algebraic variety, $p \in V$, $A = \mathcal{O}_{V,p}$, $f \in A$. If f is not a zero diviosr, then $f|_{V_i} \neq 0$, where $V = V_1 \cup \cdots \cup V_r$ and V_i is an irreducible component containing p. Otherwise, if f is not a unit, then $p \in V(f)$. Above data implies that $\dim V(f) = \dim V - 1$.

Corollary 7.23. Let V be an affine algebraic variety, if $f \in \Gamma(V, \mathcal{O}_V)$ is neither a zero divisor nor a unit, then $\dim V(f) = \dim V - 1$.

•Recall: Regular sequence and depth.

Let A be a ring, M an A-module.

(1) A sequence $x_1, ..., x_r$ of elements in A is called **regular** for M if x_i is not zero divisor for $M/(x_1, ..., x_{i-1})M$ and x_1 is not a zero divisor for M.

(2) If A is a local ring with maxiaml m, then the **depth** of M is the maxiaml length of a regular sequence $x_1, ..., x_n \in \mathfrak{m}$ for $\mathfrak{m}$.

Geometrically, if V is an irreducible affine algebraic variety, $p \in V$, $A = \mathcal{O}_{V,p}$, $f_1, ..., f_r \in \mathfrak{m}_p \subseteq \mathcal{O}_{V,p}$, imagine this process(quotient new f_i) as cut by new hypersurface (f_i), hence the 'dimension' decreases. **pic**

Example 7.24.

(1) $x_1, ..., x_r$ form a regular sequence of $k[x_1, ..., x_r]$.

(2) $x_1, ..., x_r$ form a regular sequence of $k[x_1, ..., x_r]_{m_0}$, local ring at $(0, ..., 0) \in \mathbb{A}_k^r$.

Remark 7.25. Let $(A, \mathfrak{m})$ be a local ring, then depth($\mathfrak{m}$) or write depth$(A) \leq \dim_K A$ and in general, it is strict!

Definition 7.26. Let $(A, \mathfrak{m})$ be a local Noetherian ring, A is called **Cohen-Macauly** if depth $A = \dim_K A$.

Definition 7.27. An irreducible affine algebraic variety is called **Cohen-Macauly** if all its local rings are Cohen-Macaulay.

Definition 7.28 (Complete intersection)**.** An affine algebraic variety $V \subseteq \mathbb{A}_k^n$ is called a **complete intersection** if $\exists F_1, ..., F_r \in k[x_1, ..., x_n]$ such that $I(V) = \langle F_1, ..., F_r \rangle$ and $r = n - \dim V$.

Example 7.29.

(1) A complete intersection is Cohen-Macaulay, in particular, if F is an irreducible separable polynomial in $k[x_1, ..., x_n]$, then $V(F) \subset \mathbb{A}_k^n$ is Cohen-Macaulay.

(2) All irreducible affine algebraic varieties of dim 1 is Cohen-Macaulay.

Remark 7.30. Just understand the example by realizing complete intersection and depth as accurate cut!

8 Lecture 8.

8.1 §C. Singularity and Zariski tangent space

Example 8.1. $V = V(X^3 + Y^3 - XY) \subseteq \mathbb{A}_k^n$, then in Euclidean topology, locally at $(0, 0)$, we have $V \approx$ 'cross' at $(0, 0)$ which is not smooth. V is NOT a submanifold of $\mathbb{A}_k^n$, i.e. V is singular at $(0, 0)$.

•Recall: Implict function theorem:
Let $M = V(f_1, ..., f_r) \subseteq \mathbb{R}^n$, f_i is a C^1 function.

$$J = \left(\frac{\partial f_i}{\partial x_j}\right)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}$$

Example 8.2. $f = x_1^2$, $V(f) = \{(x_1, ..., x_n) \mid x_1 = 0\}$ is a submanifold, but rank $J\mid_{V(f)} = 0$.

Remark 8.3. The strange phenomenon is due to we take too little functions.

Theorem 8.4 (Implicit function theorem)**.** *M is an m-dimensional submanifold of $\mathbb{R}^n$ $\Longleftrightarrow$: $\forall x \in M$, there exists $x \in U_x \subseteq \mathbb{R}^n$, and $g_1, ..., g_s \in C^1(U_x)$ such that $M \cap U_x = V(g_1, ..., g_s)$*

$$J(g_1, ..., g_s) = \left(\frac{\partial g_i}{\partial x_j}\right)_{\substack{1\leq i\leq s\\ 1\leq j\leq n}}$$

has rank $n - m$ over $M \cap U_x$.

Example 8.5. $\mathbb{A}_k^2 \to \mathbb{A}_k^1$ with $(X, Y) \mapsto X^3 + Y^3 - XY = F$, $J = (3X^2 - Y, 3Y^2 - X)$. See that $J(0, 0) = (0, 0)$ with $\text{rank}_{(0,0)} = 0$. And $J(x_0, y_0) \neq (0, 0)$ if and only if $(x_0, y_0) \neq (0, 0)$. Hence $J(F)$ has rank 1 over $V \setminus \{(0, 0)\}$. And $V \setminus \{0, 0\}$ is a submanifold of $\mathbb{A}_k^n$ and $(0, 0)$ is a singular point of V.

8.1.1 Singular point: non-intrinsic definition

Definition 8.6. $V \subseteq \mathbb{A}_k^n$ an affine algebraic variety. $F_1, ...F_r \in k[x_1, ..., x_n]$ are generators of $I(V)$. Let $p \in V$ be a point, then we call p a **non-singular** point if J at p

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{1\leq i\leq r\\ 1\leq j\leq n}}$$

has rank $n - m$, where $m = \dim_p V$, otherwise p is called **singular**.

Recall 8.7 (Derivative)**.** In geberal, we can NOT define the derivative of a function as that done in calculous, since there is no natural distance function on k. However we can do it formally.

Example 8.8. (1) $V = V(X^3 + Y^3 - XY) \subseteq \mathbb{A}_k^2$. $(0, 0)$ is the only singular point.

(2) $V = V(Y^2 - X^3) \subseteq \mathbb{A}_k^2$. $(0, 0)$ is the only singular point.

8.1.2 Tangent space: nonintrisic definition

tangent space of V at p = linear approximation of V at p.
= zeros of linear approximation of defining equations of V at p.
= zeros of linear parts of all $F \in I(V)$ at p.
= zeros of linear parts of generators of $I(V)$ at p.

$F \in k[x_1, ..., x_n]$, $p \in \mathbb{A}_k^n$ a point with $F(p) = 0$, then the linear part D_pF is defined as:

$$D_pF := \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(p)(x_i - p_i) \quad p = (p_1, ..., p_n).$$

it is just Taylor expansion of order 1, the **linear approximation!**

Example 8.9. $V = V(X^3+Y^3-XY) \subseteq \mathbb{A}_k^2$, namely $F = X^3+Y^3-XY$ hence $\frac{\partial F}{\partial X} = 3X^2-Y$, $\frac{\partial F}{\partial Y} = 3Y^2 - X$. $D_{(0,0)}F = 0$, $(0, 0)$ is the singular point, we draw the tangent space at $(\frac{1}{2}, \frac{1}{2})$, where $D_{(\frac{1}{2},\frac{1}{2})} = \frac{1}{4}(X - \frac{1}{2}) + \frac{1}{4}(Y - \frac{1}{2})$

Definition 8.10 (Tangent space of V at p)**.** Let $V \subseteq \mathbb{A}_k^n$ be an affine algebraic variety, $p \in V$ a point, $F_1, ..., F_r \in k[x_1, ..., x_n]$ are generators of $I(V)$. Then the **tangent space** T_pV of V at p is defined as

$$\begin{aligned} T_pV :&= V\left\{D_pF \middle| F \in I(V)\right\} \\ &= V\left\{D_pF_i \middle| 1 \leq i \leq r\right\} \subseteq \mathbb{A}_k^n \end{aligned}$$

T_pV is actually a vector space passing through p(since p always satisfies the equation).

Fact 8.11. $T_pV \rightsquigarrow T_pV - p$ is a linear subspace in $\mathbb{A}_k^n$.

Proposition 8.12 (Criterion for singularity)**.** $V \subseteq \mathbb{A}_k^n$ an affine algebraic variety, $p \in V$ is a point. Then V is non-singular at p if and only if

$$\dim T_pV = \dim_k(T_pV - p) = \dim_p V$$

see $\dim T_pV = \dim_k(T_pV - p)$, the left one is the dimension of algebraic variety, the right one is the dimension of vector space.

Proof. $p \in V$ non-singular $\iff$ $\operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}(p)\right)_{\substack{1\le i\le r\\ 1\le j\le n}} = n - \dim_p V$, where $\{F_i\}_{1\le i\le r}$ generates $I(V)$, then

$$\begin{aligned} T_pV :&= \left\{(x_1, ..., x_n) \in \mathbb{A}_k^n \middle| \sum_{i=1}^n \frac{\partial F_j}{\partial x_i}(p)(x_i - p_i) = 0, \quad 1 \le j \le r\right\} \\ T_pV - p :&= \left\{(x_1, ..., x_n) \in \mathbb{A}_k^n \middle| \sum_{i=1}^n \frac{\partial F_j}{\partial x_i}(p)x_i = 0, \quad 1 \le j \le r\right\} \\ &= \ker\left(\mathbb{A}_k^n \xrightarrow{\left(\frac{\partial F_j}{\partial x_i}(p)\right)} \mathbb{A}_k^r\right) \end{aligned}$$

$\dim(T_p - p)$ equals to $\dim_p V \iff \operatorname{rank}\left(\frac{\partial F_i}{\partial x_j}(p)\right) = n - \dim_p V$ ☐

Remark 8.13. In general, we always have $\dim_p V \le \dim T_pV$.

Definition 8.14 (Tangent bundle)**.** Let $V \subseteq \mathbb{A}_k^n$ be an affine algebraic variety, define:

$$\begin{array}{l} T_V^{Zar} := \{(p, v) \in \mathbb{A}_k^n \times \mathbb{A}_k^n \mid p \in V, v \in T_pV - p\} \\ \big\downarrow \pi \\ V \qquad \text{where } \pi \text{ is the first projection} \end{array}$$

Remark 8.15. For any $p \in V$, $\pi^{-1}(p) = T_pV - p$ is a k-vector space of dimension $\dim T_pV$ which coincides with the version of algebraic topology.

Proposition 8.16. $T_V^{Zar} \subseteq \mathbb{A}_k^n \times \mathbb{A}_k^n \overset{\text{set}}{=} \mathbb{A}_k^{2n}$ is an affine algebraic set.

Proof. Let $(x_1, ..., x_n : y_1, ..., y_n)$ be the coordinate of $\mathbb{A}_k^{2n}$, $F_1, ..., F_r \in k[x_1, ..., x_n]$ generate $I(V)$. Define

$$\widetilde{F_i} := \sum_{j=1}^n \frac{\partial F_i}{\partial x_j} y_j \in k[x_1, ..., x_n; y_1, ..., y_n]$$

Check that $T_V^{Zar} = V(F_1, ..., F_r; \widetilde{F_1}, ..., \widetilde{F_r}) \subset \mathbb{A}_k^{2n}$ ☐

Definition 8.17. $V \subseteq \mathbb{A}_k^n$, an affine algebraic varietysay V is **non-singular** if V is non-singular at any point.

9 Lecture 9.

22/9/28.

9.0.1 Zariski tangent space: intrinsic definition

$V \subseteq \mathbb{A}_k^n$ as an affine algebraic variety, $p \in V$ a point, $A(V) = k[x_1, ..., x_n]/I(V)$ coordinate ring, $\mathfrak{p} = \{f \in A(V) \mid f(p) = 0\}$ is a maximal ideal. $\forall f \in \mathfrak{p}$, choose $F \in k[x_1, ..., x_n]$ such that $f = F \mid_V$, we define a linear map:

$$d_p f : T_pV - p \subseteq \mathbb{A}_k^n = k^n \xrightarrow{\mathbf{A}} k$$
$$v = (v_1, ..., v_n) \mapsto \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(p) v_i$$

where $\mathbf{A} = \begin{pmatrix} \frac{\partial F}{\partial x_1}(p) \\ \vdots \\ \frac{\partial F}{\partial x_n}(p) \end{pmatrix}$.

Lemma 9.1. $d_p f$ is well-defined.

Proof. Choose another G such that $f = G \mid_V$, which means $F - G \in I(V)$.

$$\sum_{i=1}^{n} \frac{\partial (F-G)}{\partial x_i}(p) v_i = 0 \quad \forall (v_1, ..., v_n) \in T_pV - p$$

Hence $d_p f$ is well-defined. □

Hence we obtain a k-linear map:

$$\mathfrak{p} \overset{d_p}{\to} \mathrm{Hom}_k(T_pV - p, k)$$
$$f \mapsto d_p f$$

Proposition 9.2. The k-linear map d_p induces a k-linear isomorphism of k-vector space

$$\mathfrak{p}/\mathfrak{p}^2 \overset{d_p}{\simeq} \mathrm{Hom}_k(T_pV - p, k)$$

Proof. Step 1. d_p is surjective. Choose an embedding $\mathrm{Hom}_k(T_pV - p, k) \hookrightarrow \mathrm{Hom}_k(k^n, k)$ just like basis $(e_1, ..., e_m; e_{m+1}, ..., e_n)$. Given $L \in \mathrm{Hom}_k(T_pV - p, k)$

$$L = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} : k^n \to k$$

define $F = \sum_{i=1}^{n} c_i (x_i - p_i) \in k[x_1, ..., x_n]$, $f = F \mid_V$.

$\Rightarrow d_p f = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathrm{Hom}_k(T_pV - p, k)$

Step 2. $\ker(d_p) = \mathfrak{p}^2$. Without loss of generality, we may assume p is the origin of $\mathbb{A}_k^n$, choose $F \in k[x_1, ..., x_n]$, $f = F|_V$ such that $d_p f = 0$ i.e.

$$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i} v_i = 0 \quad \forall (v_1, ..., v_n) \in T_p V$$

$\square$

As a consequence, we obtain an isomorphism of k-vector space:

$$\mathfrak{p}/\mathfrak{p}^2 \overset{d_p}{\simeq} \mathrm{Hom}_k(T_p V, k)$$

Now we come to the intrinsic definition.

Definition 9.3 (Zariski tangent space: intrisnic definition)**.** V an affine algebraic variety, $A = \Gamma(V, \mathcal{O}_V)$, $p \in V$ a point, $\mathfrak{p}$ is the ideal of p at A, $\mathfrak{m}_p$ the maximal ideal of $\mathcal{O}_{V,p}$.

(1) The **cotangent space** $\Omega_{V,p}$ of V at p is defined as

$$\Omega_{V,p} := \mathfrak{p}/\mathfrak{p}^2 = \mathfrak{m}_p/\mathfrak{m}_p^2 \quad \text{as } k\text{-vector space}$$

(2) The **Zariski tangent space** $T_p V$ of V at p is defined as

$$T_p V := \left(\mathfrak{p}/\mathfrak{p}^2\right)^* = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*$$

Corollary 9.4. V is an affine algebraic variety, $p \in V$, V is non-singular at p if and only if $\dim_k \mathfrak{m}_p/\mathfrak{m}_p^2 = \dim_p V$.

9.0.2 Regular local ring

Let $(A, \mathfrak{m})$ be a Noetherian local ring, $k = A/\mathfrak{m}$ be the residue field, then A is called a **regular local ring** if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim_K A$, where the left one is dimension of k-vector space, the right one is Krull dimension.

Proposition 9.5. Let V be an affine algebraic variety.

(1) A point $p \in V$ is non-singular $\iff \mathcal{O}_{V,p}$ is a regular local ring.

(2) A point $p \in V$ is non-singular $\Rightarrow V$ is at p.

(3) V_{sing} is closed.

(4) $V \setminus V_{\text{sing}}$ is a dense open(only $\mathrm{Char}\, k = 0$!)

Proof. (1) By definition.

(2) Pure commutative algebra.

(3) By definition and rank of Jacobi matrix.

(4) The existence of non-singular point is NOT trivial. See [[Har77] , Chapter I, Thm 5.3].

$\square$

9.1 §D. Normality

(∗) In general, the singular locus V_{sing} of an affine algebraic variety may be very large like condimension 1!

Example 9.6. $V = V(Y^2 - X^3) \subseteq \mathbb{A}^2_k$. See that $\dim V = 1$, $V_{\text{sing}} = \{(0,0)\}$, $\dim V_{\text{sing}} = 0$. You may think a singleton is small, but in topology, it is big in this case.

Recall 9.7. Let B be a subring of A.

(1) An element $b \in B$ is **integral** over A if it satisfies an equation

$$x^k + a_1 x^{n-1} + \cdots + a_k = 0 \quad a_i \in A$$

(2) B is **integral** over A if any element $b \in B$ is integral over A.

(3) Assume further that A is an integral domain. Then A is **integrally closed** if for any element $b \in K = \mathrm{Frac}(A)$ is integral over A is in A.

Definition 9.8 (Normal variety)**.** Let V be an affine algebraic variety. Then V is called **normal** if $\mathcal{O}_{V,p}$ is integrally closed(hence domain) $\forall p \in V$.

Remark 9.9. (1) V is normal $\Rightarrow$ V is locally irreducible, i.e. $\forall p \in V$, $\exists p \in U \overset{\text{open}}{\subseteq} V$ such that U is irreducible, which means local is not a 'cross'.

Picture

(2) In other words, irreducible components are disjoint.

picture

(3) If V is non-singular, then X is normal[regular local ring is normal/integrally closed. Matsumura Prop.19.4.].

Example 9.10.

$V = V(Y^2 - X^3) \subseteq \mathbb{A}^2_k$ is not normal at $(0,0)$. See $\mathrm{Frac}(\mathcal{O}_{V,p}) \ni \left(\frac{Y}{X}\right)^2 = X \in \mathcal{O}_{V,p} \Rightarrow \frac{Y}{X}$ is integral over $\mathcal{O}_{V,p}$ but $\frac{Y}{X} \notin \mathcal{O}_{V,p}$.

Remark 9.11. See the example above, $\mathcal{O}_{V,p}$ is actually difficult to compute in general, but in this case, singular point p is $(0,0)$ which corresponds to (X,Y), hence $\mathcal{O}_{V,p}$ is just $\mathcal{O}_V$, and $\mathrm{Frac}(\mathcal{O}_{V,p})$ is easy to compute. Actually we can always move the singular point to the origin as we did in defining tangent space or computing intersection number.

Two important properties of normal varieties

Theorem 9.12 (Codimension of singular locus is greater than 2)**.** *Let V be an irreducible normal affine algebraic variety. Then $V_{sing} := \left\{p \in V \middle| V \text{ is singular at } p\right\}$ has codimension greater than 2. i.e.* $\dim V - \dim V_{sing} \geq 2$.

Proof. □

Proposition 9.13 (Partial converse. Serre)**.** Let V be an irreducible normal affine algebraic variety, such that $\operatorname{codim}_V V_{\text{sing}} \geq 2$. If V is Cohen-Macaulay, then V is normal.

Corollary 9.14. An one-dimensional irreducible normal algebraic variety is non-singular.

Recall 9.15 (Hartogs's extension theorem)**.** Let V be an irreducible normal affine algebraic variety. Let $U \subseteq V$ be an open subset such that $\dim_V(V \setminus U) \geq 2$. Then $\Gamma(V, \mathcal{O}_V) \to \Gamma(U, \mathcal{O}_U)$ is surjective by $f \mapsto f\,|_U$ i.e. regular functions on U extend to V.

Example 9.16. $f : \mathbb{C}^2 \setminus \{(0,0)\} \to \mathbb{C}$ holomorphic, then f can be extended to a global holomorphic function $\widetilde{f} : \mathbb{C}^2 \to \mathbb{C}$.

Recall 9.17 ([Mus]. Theorem 11.5.)**.** Let A be a normal($A_\mathfrak{m}$ is integrally closed over any $\mathfrak{m} \in \operatorname{MaxSpec} A$) Noetherian integral domain, then

$$A = \bigcap_{\substack{\operatorname{ht} \mathfrak{p}=1 \\ \mathfrak{p} \in \operatorname{Spec} A}} A_\mathfrak{p} \subseteq \operatorname{Frac} A$$

recall that normality is a local property, hence we only need to consider $\mathfrak{m} \in \operatorname{MaxSpec} A$.

Theorem 9.18 (Algebraic Hartog's Theorem)**.** *Let V be an irreducible normal affine algebraic variety, and $U \subseteq V$ be an open subset such that* $\operatorname{Codim}(V \setminus U) \geq 2$. *Then $\Gamma(V, \mathcal{O}_V) \to MO_U(U)$ is surjective by $f \mapsto f|_U$ i.e. regular functions on U can extend to V.*

Proof. Choose $s \in \Gamma(U, \mathcal{O}_U)$, $\forall\ Y \subseteq X$ an irreducible subvariety of codimension 1, then $Y \cap U \neq \varnothing$. Hence $\exists f, g \in \Gamma(V, \mathcal{O}_V)$ such that $s = \frac{f}{g}$ and $g|_Y \neq 0$, and thus $s \in A_{I(Y)}$, where $I(Y) = \{f \in \Gamma(V, \mathcal{O}_V) | f_Y = 0\} \subseteq \Gamma(V, \mathcal{O}_V)$ is a prime ideal with height 1.

$$\{\text{irreducible closed subset of codimension 1 in } V\} \overset{1:1}{\longleftrightarrow} \{\text{prime ideals of height 1 in } \Gamma(V, \mathcal{O}_V)\}$$

$\overset{\text{Matsumura}}{\Longrightarrow} s \in A \subseteq \operatorname{Frac}(A) \Rightarrow s \in \Gamma(V, \mathcal{O}_V)$, hence $A = \Gamma(V, \mathcal{O}_V)$. □

9.1.1 Normalization of an affine algebraic variety

Theorem 9.19 (Normalization)**.** *Let V be an irreducible affine algebraic variety. Then there exists an irreducible affine algebraic variety V^{nor} with a morphism $n : V^{nor} \to V$ such that*

(1) V^{nor} is normal.

(2) $n^{\#} : \Gamma(V, \mathcal{O}_V) \to \Gamma(V^{nor}, \mathcal{O}_{V^{nor}})$ *is integrally closed, i.e.* $\Gamma(V^{nor}, \mathcal{O}_{V^{nor}})$ *is integrally over* $\Gamma(V, \mathcal{O}_V)$. *We call* $n : V^{nor} \to V$ *the normalization of* V.

Example 9.20. $V = V(Y^2 - X^3) \subseteq \mathbb{A}^2_k$. Then the normalization of V is

$$\begin{aligned} V^{nor} = V(T - W^2) &\xrightarrow{n} V \\ (T, W) &\mapsto (T, TW) \end{aligned}$$

see $V^n \simeq \mathbb{A}^1_k$.

10 Lecture 10.

22/10/10.

Theorem 10.1 (Normalization)**.** *let* V *be an irreducible affine algebraic variety. Then there exists an irreducible affine algebraic variety* V^{nor} *with a morphism*

$$n : V^{nor} \to V$$

such that

(1) V^{nor} *is a normal variety.*

(2) $n^{\#}_V : \Gamma(V, \mathcal{O}_V) \to \Gamma(V^{nor}, \mathcal{O}_{V^{nor}})$ *induces an isomorphism of fields*

$$\mathrm{Frac}(\Gamma(V, \Gamma(V, \mathcal{O}_V))) \simeq \Gamma(V^{nor}, \mathcal{O}_{V^{nor}})$$

(3) $\Gamma(V^{nor}, \mathcal{O}_{V^{nor}})$ *is a finite* $\Gamma(V, \mathcal{O}_V)$*-module.*

Example 10.2. $V = V(Y^2 - X^3) \subseteq \mathbb{A}^2_k$. The normalization of V is $V^{nor} = V(T - W^2) \subseteq \mathbb{A}^2_k$ with

$$\begin{aligned} n : V^{nor} &\to V \\ (T, W) &\mapsto (T, TW) \end{aligned}$$

Remark 10.3.

(a) (2) means n is **birational**, i.e. $\exists U \subseteq V^{nor}$ and $U' \subseteq V$ open subsets such that

$$n|_U : U \to U'$$

is an isomorphism.

(b) (3) means that n is **finite** which means fibres of n is finite(but not necessary conversely).

Proof. Take $A =$ integral closure of $\Gamma(V, \mathcal{O}_V)$, by Noetherian normalization theorem, there exists a subring

$$B = k[T_1, ..., T_m] \subseteq \Gamma(V, \mathcal{O}_V)$$

such that $\Gamma(V, \mathcal{O}_V)$ is integral over B. Then

Diagram

(1) A is the integral closure of B in $K(V)$ by transitivity of integral.

(2) B is integrally closed in Frac(B). Since $\Gamma(\mathbb{A}_k^m, \mathcal{O}_{\mathbb{A}_k^m}) = B$, $\mathbb{A}_k^n$ is nonsingular hence $\mathbb{A}_k^m$ is normal which means B is integrally closed. $\Rightarrow$: A is a finite B-module [AM. Prop.5.17.]. $\Rightarrow$: A is a finite $\Gamma(V, \mathcal{O}_V)$-module. $\Rightarrow$: A is a finitely generated k-algebra and an integral domain, so

$$A \simeq k[Y_1, ..., Y_N]/I$$

I is a prime ideal. We take $V^{nor} = V(I) \subseteq \mathbb{A}_k^N$ and $n : V^{nor} \to V$ is induced by $\Gamma(V, \mathcal{O}_V) \hookrightarrow \Gamma(V^{nor}, \mathcal{O}_{V^{nor}}) = A$.

□

Recall 10.4. Let $A \to B$ be Noetherian rings such that B is a finite A-module. Then for any maximal ideal $\mathfrak{m}$ of A, $\mathfrak{m} \cdot B \subset \mathfrak{m}'$ for some maximal ideal $\mathfrak{m}' \subseteq B$. Hence $n : V^{nor} \to V$ is surjective.

Definition 10.5. Let V be an affine algebraic variety(maybe reducible.). Then the **normalization** of V is the disjoint union of the normalization of each irreducible components.

Example 10.6. If we have $V = (XY) \subset \mathbb{A}_k^2$, we know the normalization of a single line is itself, hence the normalization of V is just the disjoint union of two lines, we 'disjoint' them like blow-up.

V. General Algebraic Varieties

Notation 10.7.

(1) Differential manifold = a Hausdorff locally ringed space $(X, \mathcal{O}_X)$ such that $\forall\ x \in X$, there exists a neighborhood U_x of x such that $(U_x, \mathcal{O}_X|_{U_x}) \simeq \big(B(0,r), \mathcal{O}_{B(0,r)}\big)$.

(2) We hope to define a general algebraci variety to be a 'Hausdorff' locally ringed space such that $\forall\ x \in X$ there exists a neighborhood U_x of x such that $(U_x, \mathcal{O}_X|_{U_x}) =$ affine algebraic variety. <u>BUT</u>: Zariski topology is never Hausdorff.

10.1 §A. Prevarieties

Definition 10.8 (Prevariety)**.** A **prevariety** over k is a locally ringed space $(X, \mathcal{O}_X)$ such that

(1) X is quasi-compact.

(2) $\forall\ x \in X$, there exists U_x open neighborhood of x such that $(U_x, \mathcal{O}_X|_{U_x}) \simeq$ some affine algebraic variety.

Definition 10.9. Let $(X, \mathcal{O}_X)$ be a prevariety, $U \subseteq X$ an open subset. Then $(U, \mathcal{O}_X|_U)$ is a prevariety, called an **open subvariety** of X.

Remark 10.10. Then $X = \cup_{i=1}^n X_i$ where X_i = affine algebraic variety and $U_i = U \cap X_i$ is a finite union of affine open subsets of X_i.

An open subset of an affine algebraic variety is a prevariety called **quasi-affine variety**.

Example 10.11. Quasi-affine variety maybe not affine!
$U = \mathbb{A}_k^2 \setminus \{(0,0)\}$ is quasi-affine, but not affine.
Consider

$$i^{\#} : \Gamma(\mathbb{A}_k^2, \mathcal{O}_{\mathbb{A}_k^2}) \to \Gamma(U, \mathcal{O}_U)$$

which is induced by inclusion. By algebraic Hartog's theorem, $i^{\#}$ is an isomorphism(since a point has codimension 2 in this plane case.). But it is absurd since affine variety is one to one correspongding to its global section. What's more, we can compute its cohomology and it does not vanish in higher dimensional.

Lemma 10.12. A prevariety is Noetherian. In particular, it admits a unique decomposition into irreducible components.

Proof. Our whole space is X, assume $Y_1 \supseteq Y_2 \supseteq \cdots$ is a sequence of closed subsets. Let $U = \cup_{i=1}^{\infty}(X \setminus Y_i) \subset X$, each $X \setminus Y_i$ is open, U is quasi-compact. Then, there there exists $m \geq 1$ such that $U = \cup_{i=1}^{m}(X \setminus Y_i) \Rightarrow Y_i = Y_m$ for all $m \geq i$. □

Definition 10.13 (Closed subprevariety)**.** $(X, \mathcal{O}_X)$ is a prevariety, $i : Z \to X$ is a proper closed subset. Then $(Z, \mathcal{O}_Z)$ is a prevariety called a **closed subprevariety** if $\Gamma(U, \mathcal{O}_U) =$

$$\{f : U \to K : \forall x \in u,\ \exists \text{ open neighborhood } U_x \text{ and } g \in \Gamma(U_x, \mathcal{O}_X) \text{ such that } f|_{U_x \cap U} = g|_{U_x \cap U}\}$$

11 Lecture 11.

22/10/12.

Recall 11.1 (Locally closed subset)**.** X is a topological space, $Z \subseteq X$ is **locally closed** if $Z = U \cap Y$ where $U \subset X$ open and $Y \subseteq X$ closed.

Definition 11.2 (Subvariety)**.** Let X be a prevariety, $Z = U \cap Y$ be a locally closed subset with structure sheaf $\mathcal{O}_Z$, a closed subvariety of U is called a **subprevariety** of X, i.e. a closed in open or an open in closed, lol.

Remark 11.3 ($\star$)**.** Let X be a prevariety. Then

(1) $U \hookrightarrow X$ open, $\mathcal{O}_U = i^{-1}\mathcal{O}_X$.

(2) $Z \hookrightarrow X$ closed, $\mathcal{O}_Z \neq i^{-1}\mathcal{O}_X$.

11.1 §B. Separateness and Varieties

Example 11.4. $U_1 = (\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})$, $U_2 = (\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})$.
Define $X = (U_1 \bigsqcup U_2)/\sim$, where $x \sim y \iff x = y$ if $x, y \neq 0$, just glue to an affine line with double original point.

Definition 11.5 (separateness)**.** Let X be a prevariety.

(1) We say that X is **separate** if all prevarieties Y and all morphisms

$$Y \underset{g}{\overset{f}{\rightrightarrows}} X$$

the set $\{y \in Y : f(y) = g(y)\}$ is closed in Y.

(2) A **variety** is a separate prevariety.

Back to our previous example: Take

$$Y \underset{g}{\overset{f}{\rightrightarrows}} X$$

with f mapping to U_1 indentically and g mapping to U_2 indentically.
Then $\{z \in \mathbb{A}^1_k : f(z) = g(z)\} = \mathbb{A}^1_k \setminus \{(0,0)\}$ is not a closed subset of Y, hence X is not separate.

Remark 11.6 (Geometric meaning of separateness = limit is unique!)**.** Assume X is NOT separate, then there exists Y prevariety with morphisms

$$Y \underset{g}{\overset{f}{\rightrightarrows}} X$$

such that $S := \{y \in Y : f(y) = g(y)\}$ is not closed, hence $\exists \{y_k\} \subset S$ such that $y_k \to y \notin S$, which means $f(y) \neq g(y)$ but $\{x_k = f(y_k) = g(y_k)\} \to f(y)$ and $g(y)$.

11.2 §C. Products of Prevarieties: A Criterion for Separateness

(a) Product of Affine Algebraic Varieties

Let $X \subset \mathbb{A}_k^n$, $Y \subseteq \mathbb{A}_k^m$ be two irreducible affine varieties. The product of X and Y, denoted by $X \times Y$ is an irreducible affine algebraic variety:

(1) As a set, $X \times Y = \{(x, y) \in \mathbb{A}_k^n \times \mathbb{A}_k^m = \mathbb{A}_k^{m+n} : x \in X,\ y \in Y\}$.

(2) The Zariski topology of $X \times Y$ is the subspace topology induced from $\mathbb{A}_k^{m+n}$(not the product topology of $\mathbb{A}_k^n$ and $\mathbb{A}_k^m$.).

(3) The structure sheaf $\mathcal{O}_{X \times Y}$ = structure sheaf of $X \times Y$ as an affine algebraic set in $\mathbb{A}_k^{m+n}$.

Lemma 11.7. $X \times Y \subset \mathbb{A}_k^{m+n}$ is an affine algebraic set, it is obvious since we can embed their functions into higher-dimensional space.

Lemma 11.8. $\Gamma(X \times Y, \mathcal{O}_{X \times Y}) \simeq \Gamma(X, \mathcal{O}_X) \otimes \Gamma(Y, \mathcal{O}_Y)$.

Proof.

$$\begin{aligned}
\Gamma(X, \mathcal{O}_X) \otimes \Gamma(Y, \mathcal{O}_Y) &= \frac{k[X_1, ..., X_m]}{(F_1, ..., F_r)} \otimes_k \frac{k[Y_1, ..., Y_n]}{(G_1, ..., G_s)} \\
&= \frac{k[X_1, ..., X_m; Y_1, ..., Y_n]}{(F_1, ..., F_r; G_1, ..., G_s)}
\end{aligned}$$

recall that the tensor product of two domains is still a domain, see [[ZS75]. Chapter 3. §15]. □

This tells us that $X \times Y$ can be defined intrinsically!

(b) Product of Prevarieties

Let $X = \cup_{i=1}^m X_i$, $Y = \cup_{j=1}^n Y_j$ with X_i, Y_j be irreducible open affine algebraic sets.

Definition 11.9. The product of X and Y, denoted by $X \times Y$, is a prevariety such that

(1) $X \times Y = \{(x, y) : x \in X,\ y \in Y\}$ as a set.

$\Rightarrow X \times Y = \bigcup_{i,j} X_i \times Y_j$ as a set.

(2) The Zariski topology on $X \times Y$ is the topology induced by the Zariski topology on $X_i \times Y_j$ i.e.

$$U \subseteq X \times Y \text{ open } \iff \left[\begin{array}{l} (1)\ X_i \times Y_j \text{ is open}, \forall i, j \\ (2)\ U \cap (X_i \times Y_j) \text{ is open in } X_i \times Y_j\ \forall i, j \end{array} \right]$$

(3) The structure sheaf $\mathcal{O}_{X \times Y}$ is the unique sheaf on $X \times Y$ such that$\mathcal{O}_{X \times Y}|_{X_i \times Y_j} = \mathcal{O}_{X_i \times Y_j}$.

Remark 11.10. Since $X = \cup_{i=1}^m X_i$, we can refine it into a 'smaller' cover, just refine each affine open to an affine open cover by $D(f)$.

Lemma 11.11. Let X and Y be two irreducible affine algebraic varieties. Take $0 \neq f \in \Gamma(X, \mathcal{O}_X)$ and $0 \neq g \in \Gamma(Y, \mathcal{O}_Y)$. Then

$$\mathcal{O}_{X\times Y}\big|_{D(f)\times D(g)} \simeq \mathcal{O}_{D(f)\times D(g)}$$

see that $D(f)$ and $D(g)$ are irreducible affine algebraic varieties since they are localization of domains.

Proof. $D(f) \times D(g) = D(fg) \subseteq X \times Y$. □

Now $(X_i \times Y_j) \cap (X_i' \times Y_j') = U \subseteq X \times Y$.

Question 11.12. $\mathcal{O}_{X_i\times Y_j}\big|_U \overset{?}{=} \mathcal{O}_{X_i'\times Y_j'}\big|_U$

We make refinement: $X_i \cap X_i' = \cup X_i''$ with X_i'' are all affine. Then

$$\mathcal{O}_{X_i\times Y_j}\big|_{X_i''\times Y_j''} \simeq \mathcal{O}_{X_i'\times Y_j'}\big|_{X_i''\times Y_j''}$$

hence we localize to an affine open cover which gives a unique glue.

(2) Criterion for Seperatedness

Proposition 11.13. Let X be a prevariety. Consider the diagonal morphism

$$X \overset{\Delta}{\to} X \times X$$
$$x \mapsto (x, x)$$

then X is separate if and only if $\Delta(X)$ is closed in $X \times X$.

Proof. $\Rightarrow$: let $p_i : X \times X \to X$ for $i = 1, 2$ and $\Delta(X) = \{(x, x) \in X \times X : p_1(x, x) = p_2(x, x)\}$, hence $\Delta(X)$ is closed since X is separate.
$\Leftarrow$: assume $\Delta(X)$ is closed, take an arbitray prevariety Y with $f, g : Y \to X$.

$$\begin{aligned} S := &\{y \in Y : f(y) = g(y)\} \\ = &\Phi^{-1}(\Delta(X)) \end{aligned}$$

where $\Phi : Y \to X \times X$ by $y \mapsto (f(y), g(y))$ is closed. □

Corollary 11.14. All affine algebraic varieties are separate, moreover affine schemes are separate.

Proof. $\mathbb{A}^n_k \times \mathbb{A}^n_k = \mathbb{A}^{2n}_k$ as prevarieties, then

$$\Delta(\mathbb{A}^n_k) = V\{X_i - Y_i : 1 \leq i \leq n\}$$

which is closed. □

Corollary 11.15 (Criterion for separateness)**.** Let X be a prevariety. Assume for any x, $x' \in X$, if there there exists an open subset U containing x and x', furthermore, U is an affine algebraic variety. Then X is separate.

Proof. Y a prevariety with $f, g : Y \to X$. $Z = \{y \in Y : f(y) = g(y)\}$. Assume on the contrary that $Z \neq \overline{Z}$ in Y. Let $z \in \overline{Z}$ such that$f(z) \neq g(z)$.
Let U be an affine algebraic set of X containing $f(z)$ and $g(z)$ and

$$V = f^{-1}(U) \cup g^{-1}(U)$$

Consider $g|_V, f|_V : V \to U$, then

$$\{y \in V : f(y) = g(y)\} = Z \cap V$$

is closed in V since U is separate. But $z \in \{y \in V : f(y) = g(y)\} = Z \cap V$, a contradiction. ☐

12 Lecture 12.

22/10/17.

Remark 12.1. In the definition of product of affine algebraic varietiesin the last class, we always assume that the affine algebraic varietiesare irreducible and choose an irreducible affine open cover. In general, it is false! For example the 'corss' does not have an affine open at the origin. However, the Irreducibility is only used to ensure the tensor product of integral domians is also reduced.

Theorem 12.2 ([?] V.§15)**.** *Let* $k = \overline{k}$, $\operatorname{char} k = 0$ *and* A, B *are reduced* k*-algebra. Then* $A \otimes_k B$ *is a reduced* k*-algebra.*

12.1 §D. Completeness

Completeness = 'compactness', limit of convergent sequence always exists.
When we say 'variety', we mean a separate variety.

Definition 12.3 (Complete)**.** Let X be a variety. X is called **complete** if for any variety Y, the projection $X \times Y \xrightarrow{p} Y$ is closed, which means mapping closed subset to closed subset.

Example 12.4. $X = V(xy - 1) \subseteq \mathbb{A}_k^2$, we have natural projection: $\mathbb{A}_k^2 \times \mathbb{A}_k^1 \xrightarrow{p_2} \mathbb{A}_k^1$. Consider $X' = V(xy - 1, x - z) \subseteq \mathbb{A}_k^2 \times \mathbb{A}_k^1$ which is a closed subset, but $p_2(X') = \mathbb{A}_k^1 \setminus \{0\}$ is open. Affine space is **never** complete since it misses the infinite points, in this case, it miss $V(x)$ and $V(y)$, you can find them in $\mathbb{P}_k^2$. An affine algebraic variety is complete if and only if it has only finite points, hence

$$\mathbb{A}_k^n \text{ is complete} \iff n = 0(\mathbb{A}_k^0 \text{ is singleton})$$

Remark 12.5 (Geometric meaning of completeness)**.** Let C be a curve, $o \in C$, a marked point. Consider a morphism

$$f : C \setminus \{o\} \to X$$

and the graph

$$\Gamma(f) = \{(c, f(c)) : c \in C \setminus \{o\}\}$$

take the Zariski closure $\overline{\Gamma(f)}^{Zar}$. Consider the projection

$$C \times X \xrightarrow{p_1} C$$

$$\begin{aligned} X \text{ is complete } &\Rightarrow p_1(\overline{\Gamma(f)}^{Zar}) \text{ is closed in } C. \\ &\Rightarrow p_1(\overline{\Gamma(f)}^{Zar}) = C. \\ &\Rightarrow p_1^{-1}(o) \cap \overline{\Gamma(f)}^{Zar} \neq \varnothing. \end{aligned}$$

see $C \setminus \{o\}$ is contained in $\overline{\Gamma(f)}^{Zar}$. In geometry, the limit point is in our closure!

Proposition 12.6 (Basic properties of completeness)**.**

(1) Let $f : X \to Y$ be a morphism of varieties. If X is complete, then $f(X)$ is closed and again complete, closed map and image is complete(like image of compact is compact.).

(2) If X and Y are complete, then so does $X \times Y$.

(3) If X is complete and $Y \subseteq X$ is a closed subvariety, then Y is complete.

(4) Affine algebraic variety if complete $\iff$ its dimension is 0(hence finite points).

Remark 12.7 (Basic notions for variety)**.** Let X be a variety.

(1) Dimension of X = its topological dimension.

(2) Irreducibility = irreducible in Zariski topology.

(3) A point p in X is nonsingular if $\mathcal{O}_{X,p}$ is a regular local ring, hence $\dim_K \mathcal{O}_{X,p} = \dim_k \mathfrak{m}_p/\mathfrak{m}_p^2$.

(4) Zariski tangent space of $p \in X = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*$.

(5) X is normal if $\mathcal{O}_{X,p}$ is integrally closed.

Remark 12.8 (Compare compactness and completeness for $k = \mathbb{C}$)**.** Let X be an affine algebraic variety over k, $X = \cup X_i$, each X_i is an affine open, so we have $X_i \hookrightarrow \mathbb{A}_k^{n_i}$, since $\mathbb{A}_k^{n_i}$ has natural Euclidean topology, hence we can induce Euclidean topology on X_i, denoted by X_i^{an}(analytic) which $\rightsquigarrow X^{an}$ with Euclidean topology on X.

$$\text{Zariski topology on } X \text{ is complete} \iff X^{an} \text{ is compact in Euclidean.}$$

12.2 §E. Function Field and Rational Map

(1) Function Field

Definition 12.9 (Function field)**.** Let X be an irreducible affine algebraic variety. Then the **function field** $K(X)$ is defined as

$$K(X) = \varinjlim_{\varnothing \neq U \subseteq X} \Gamma(U, \mathcal{O}_X)$$

where U is open in X, i.e.

$$K(X) = \{(s, U) : s \in \Gamma(U, \mathcal{O}_X),\ \varnothing \neq U \subseteq X \text{ open}\} \big/ \sim$$

where $(s, U) \sim (s', U') \iff \exists \varnothing \neq W \subseteq U \cap U'$, W is open in X such that $s|_W = s'|_W$, see $U \cap U' \neq \varnothing$ since X is irreducible. And we denote the equivalence class by $\overline{(s, U)}$.

Proposition 12.10.

(1) The canonical map $\Gamma(U, \mathcal{O}_X) \hookrightarrow K(X)$ is an injective homomorphism of rings for any open U.

(2) For any nonempty open $U \subseteq X$, there exists canonical isomorphism of fields

$$K(U) \simeq K(X)$$

hence we get a criterion of birationalness.

(3) For any $x \in X$, there exists a canonical isomorphism of fields

$$\mathrm{Frac}(\mathcal{O}_{X,x}) \simeq K(X)$$

Proof. We just list the canonical morphism, and left the readers to check.

(1) $s \mapsto \overline{(s, U)}$.

(2) $\overline{(s', U')} \mapsto \overline{(s', U')}$.

(3) $\mathcal{O}_{X,x} \to K(X)$ by $\overline{(s, U)} \mapsto \overline{(s, U)}$, see it only gives the homomorphism of 'numerator', and easy to generlize to the fraction field.

□

Remark 12.11.

(1) Morphisms between varieties = morphisms of locally ringed spaces. Then f induces a morphism of fields.

$$\begin{aligned} K(Y) &\to K(X) \\ \overline{(s, U)} &\mapsto \overline{(s \circ f, f^{-1}(U))} \end{aligned}$$

Proposition 12.12. $\dim_t X = \operatorname{tr.deg}_k K(X)$. In particular, $\dim X \times Y = \dim X + \dim Y$, where $\dim_t$ means the topological dimension.

Proof. We claim that for $\varnothing \neq U \subseteq X$, then $\dim U = \dim_x U = \dim_K \mathcal{O}_{U,x}$ for any $x \in U$, where U is open in X. Let's prove this claim.

(i)

$\square$

(2) Rational Map

Definition 12.13 (Rational map)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two varieties, A **rational map**

$$f : X \dashrightarrow Y$$

is a morphism $f : (U, \mathcal{O}_X|_U) \to (Y, \mathcal{O}_Y)$ of varieties, where $(U, \mathcal{O}_X|_U)$ is an open subvariety of X.

Example 12.14.

(1) Cremona map

$$\mathbb{A}_k^2 \overset{f}{\dashrightarrow} \mathbb{A}_k^2$$
$$(x_1, x_2) \mapsto (\frac{1}{x_1}, \frac{1}{x_2})$$

f is well-defined on $D(x_1x_2)$ and induces an isomorphism

$$D(x_1x_2) \simeq D(y_1y_2)$$

inverse is given by

$$(y_1, y_2) \mapsto (\frac{1}{y_1}, \frac{1}{y_2})$$

recall that $D(x_1) \cap D(x_2) = D(x_1x_2)$.

(2) Projection

$$\mathbb{A}_k^2 \overset{p}{\dashrightarrow} \mathbb{A}_k^1$$
$$(x_1, x_2) \mapsto (\frac{x_1}{x_2})$$

p is well-defined on $D(x_2)$.

(3) $x_1x_2 - x_3x_4$

Let $X = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}_k^4$, take

$$X \overset{f}{\dashrightarrow} \mathbb{A}_k^1$$
$$(x_1, x_2, x_3, x_4) \mapsto \frac{x_1}{x_3}$$

see that $\frac{x_1}{x_3} = \frac{x_4}{x_2}$, hence f is NOT well-defined on $D(x_2)$, BUT well-defined on $D(x_2) \cup D(x_3) \subseteq X$. However, does $D(x_2) \cup D(x_3)$ is the biggest open subset such that f is well-defined on it?

The last question of above example is a fundamental question in birational geometry, how to find the maximal open subset such that it extends the rational map we have.

Now, we arrive at a conclusion about separateness and completeness:

(1) Separateness means if the limit exists, then it is unique.

(2) Completeness means separateness and the limit always exists!

Let's translate the conclusion above into [Har77] version, it corresponds to the **Valuation Criterion**, separateness means if it exists, it is unique, and properness means it exists.

13 Lecture 13.

22/10/19.

Remark 13.1. Two rational maps (f_1, U_1) and (f_2, U_2) are considered equal if $U_1 \cap U_2 \neq \varnothing$ and

$$f_1|_{U_1\cap U_2} = f_2|_{U_1\cap U_2}$$

Definition 13.2 (Locus of indeterminacy)**.** Let (f, U) be a rational map $X \dashrightarrow Y$. The **locus of indeterminacy** of f is the smallest closed subset $Z \subseteq X$ such that for any $x \in X \setminus Z$, there exists a rational map (f', U') equal to (f, U) and $x \in U$, see the existence follows from the Noetherian property of X and $X \setminus Z$ is nonempty(which corresponds to trivial case) since we can take $Z = X \setminus U$.

Remark 13.3. Let $(f, U) : X \dashrightarrow Y$ be a rational map between two irreducible varieties. Then f naturally induces a homomorphism of fields

$$f^* : K(Y) \to K(X)$$

by

$$\overline{(s, V)} \mapsto \overline{(s \circ f, f^{-1}(V) \cap U)}$$

recall that we are talking about sheaf of regular functions, so we have $s \circ f$.

Definition 13.4 (Graph of rational maps)**.** Let (f, U) be a rational map. $f : X \dashrightarrow Y$ between irreducible varieties.

(1) The **graph** of f is the closure of the graph of (f, U)

$$\Gamma(f) = \overline{\Gamma(f, U)}^{Zar} \subseteq X \times Y$$

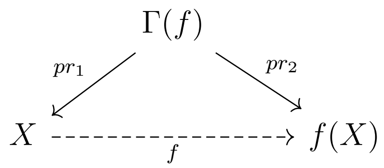
where $\Gamma(f, U) = \{(x, y) \in X \times Y : f(x) = y\} \subseteq X \times Y$

(2) The **image** of f is defined as the image of $\Gamma(f)$ under the second projection, i.e.

$$f(X) := pr_2(\Gamma(f))$$

where pr_2 is the natural projection which is not the usual image in general.

Remark 13.5. $f(X) \neq \overline{f(U)}^{Zar}$ in general, actually $f(X) \subseteq \overline{f(U)}^{Zar}$ and $f(X)$ is <u>NOT</u> necessarily closed, see that $\overline{f(X)}^{Zar} = \overline{f(U)}^{Zar}$.
$f(X) = \overline{f(U)}^{Zar}$ when X is complete. Actually, what matters is the diagram:



Definition 13.6 (Birational map)**.** Let X and Y be two irreducible varieties. A **birational map** $f : X \dashrightarrow Y$ is a rational map which is *dominant* and $f^* : K(Y) \to K(X)$ is an isomorphism.

Proposition 13.7. Let $f : X \dashrightarrow Y$ be a rational map. Then f is birational if and only if there exists $U \subseteq X$ and $V \subseteq Y$ nonempty open subsets, such that

$$f|_U : U \to V$$

is an isomorphism.

Example 13.8.

(1) (Cremona map)

$$\mathbb{A}^2_k \dashrightarrow \mathbb{A}^2_k$$
$$(x_1, x_2) \mapsto (\frac{1}{x_1}, \frac{1}{x_2})$$
$$(\frac{1}{y_1}, \frac{1}{y_2}) \leftarrow (y_1, y_2)$$

(2) Normalization of affine algebraic varieties are birational maps.

Chapter VI. Projective Varieties

13.1 §A. Projective Space

Definition 13.9. The n-dimensional **projective space** over k, denoted by $\mathbb{P}_k^n$ is the equivalence classes

$$(\mathbb{A}_k^{n+1} \setminus \{0\})/\sim$$

where $(x_0, ..., x_n) \sim (y_0, ..., y_n) \iff \exists \lambda \in k^\times$ such that $x_i = \lambda y_i$ for all i. We write $[x_0 : \cdots x_n]$ for the equivalence class, and call it homogeneous coordinate.

Example 13.10.

$(n = 0)$. $\mathbb{P}_k^0$ is a singleton $[1]$.

$(n = 1)$. $\mathbb{P}_k^1$ glues the antipoints of S^1.

$(n = 2)$. We decompose it into affine pieces. Define

$$U_0 = \left\{[1 : x_1 : x_2] \in \mathbb{P}_{\mathbb{R}}^2\right\} \simeq \mathbb{R}^2$$

Let

$$\begin{aligned} L_\infty :&= \mathbb{P}_{\mathbb{R}}^2 \setminus U_0 \\ &\{[0 : 1 : x_2] : x_2 \in \mathbb{R}\} \cup \{[0 : 0 : 1]\} \end{aligned}$$

see that $\{[0 : 1 : x_2] | x_2 \in \mathbb{R}\} \simeq \mathbb{R}$(I don't know what's the $\simeq$ means in this case, just in set no other structure?)

Now, let's explain the old saying that two parallel lines on projective space meet at the infinity point but it is not a strict proof, it is our intuition.
Let $L_1 = az + by + c_1$ and $L_2 = az + by + c_2$ with $c_1 \neq c_2$.
Choose $(z_1, y_1) = P_1 \in L_1$.

If $b \neq 0$.

$$\begin{aligned} \lim_{\substack{P_1 \to \infty \\ P_1 \in L_1}} P_1 &= \lim_{z_1 \to \infty} [1 : z_1 : y_1] \\ &= \lim_{z_1 \to \infty} \left[1 : z_1 : \frac{-az_1 - c_1}{b}\right] \\ &= \left[0 : 1 : -\frac{a}{b}\right] \end{aligned}$$

which is the slope of L_1.

Similiarly

$$\lim_{\substack{P_2 \to \infty \\ P_2 \in L_2}} P_2 = \left[0 : 1 : -\frac{a}{b}\right] \in L_\infty$$

hence

$$L_1 \cap L_2 = \left\{\left[0 : 1 : -\frac{a}{b}\right]\right\} \in L_\infty \subseteq \mathbb{P}_{\mathbb{R}}^2$$

If $b = 0$, then $a \neq 0$.

$$\begin{aligned}\lim_{\substack{P_1 \to \infty \\ P_1 \in L_1}} P_1 &= \lim_{y_1 \to \infty} [1 : z_1 : y_1] \\ &= \lim_{y_1 \to \infty} \left[1 : \frac{-by_1 - c_1}{a} : y_1\right] \\ &= [0 : 0 : 1]\end{aligned}$$

Similiar for L_2, hence

$$L_1 \cap L_2 = \{[0 : 0 : 1]\}$$

We get a beautiful describtion for L_∞, the infinite far points.

Proposition 13.11. L_∞ = the limits of lines in $\mathbb{R}^2$ at infinite such that all parallel lines meet at a unique point at L_∞.

Definition 13.12 (Projective subspaces)**.** $\mathbb{P}^n_k$, n-dimensional projective space. Let $F \subseteq \mathbb{A}^{n+1}_k$ be a linear subspace. Then the canonical image of $F \setminus \{0\}$ in $\mathbb{P}^n_k$ is called a **projective subspace** of $\mathbb{P}^n_k$, just use orinary linear subspace to cut projective space, then you get projective subspace.

13.2 §B. Zariski Topology on Projective Space

(1) Quotient topoplogy

Let $\pi : \mathbb{A}^{n+1}_k \setminus \{0\} \to \mathbb{P}^n_k$.

Definition 13.13. The **Zariski topology** on $\mathbb{P}^n_k$ is the quotient topology of the Zariski topology on $\mathbb{A}^{n+1}_k \setminus \{0\}$, i.e. $Z \subseteq \mathbb{P}^n_k$ is closed if and only if $\pi^{-1}(Z)$ is closed.

Example 13.14. Projective subspace is closed.

(2) Homogeneous ideals

Let $I \subseteq R = k[x_0, x_1, ..., x_n]$ be a homogeneous ideal. Then by Hilber basis theorem

$$I = (F_1, ..., F_r)$$

with each F_i is a homogeneous element. Then

$$\begin{aligned}V(I) &= \{[x_{0,:\cdots:x_n}] \in \mathbb{P}^n_k | F_i(x_0, ..., x_n) = 0\} \\ &= \{[x_0, ..., x_n] \in \mathbb{P}^n_k | F(x_0, ..., x_n) = 0 \quad \forall F \in R_m \cap I\}\end{aligned}$$

Remark 13.15. F_i is <u>NOT</u> a well-defined function on $\mathbb{P}^n_k$! but its zeros are well-defined.

Proposition 13.16 (Exercise)**.** $k = \overline{k}$ and $\mathrm{char}(k) = 0$.

(1) $V(R) = \varnothing$.

(2) $V(0) = \mathbb{P}^n_K$.

(3) $I \subseteq J$ are homogeneous ideals. Then $V(J) \subseteq V(I)$.

(4) $\{I_\alpha\}$ is a family of homogeneous ideals. Then

$$\bigcap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$$

(5) I, J are homogeneous ideals. Then

$$V(I) \cup V(J) = V(I \cap J)$$

Remark 13.17. The propositions above show that the subset of $\mathbb{P}^n_k$ of the form $V(I)$, where I is a homogeneous ideal, forms a topology on $\mathbb{P}^n_k$.

Proposition 13.18. Let $Z \subseteq \mathbb{P}^n_k$ be a closed subset in the Zariski topology. Then $Z = V(I)$ for some homogeneous ideal $I \subseteq k[x_0, ..., x_n] = R$.

Corollary 13.19. The Zariski topology on $\mathbb{P}^n_k$ is the same as the topology on $\mathbb{P}^n_k$ defined by homogeneous ideals.

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Remark 14.1. That variety Y should be $\pi^{-1}(V(I)) \subseteq \mathbb{A}^{n+1}_k \setminus \{0\}$.

Let's come to the third definition of topology.
(3) Covering $\mathbb{P}^n$ by $\mathbb{A}^n_k$.
For $0 \le i \le n$, define

$$U_i = \{[x_0 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n]\} \subseteq \mathbb{P}^n$$

which is open since $U_i = \mathbb{P}^n \setminus V(x_i)$.
We will see that U_i is isomorphic to $\mathbb{A}^n_k$, hence we get an affine piece. Define

$$\varphi_i : \mathbb{A}^n_k \to U_i$$
$$(y_1, ..., y_n) \mapsto [y_1 : \cdots : 1 : \cdots : y_n]$$

where '1' is in the i-th term, see φ_i is bijective.

Proposition 14.2. $\varphi_i : \mathbb{A}^n_k \to U_i$ is a homeomorphism on Zariski topology.

Proof. Without loss of generality, we may assume $i = 0$, then we have a diagram:

$$\begin{array}{ccc} & & \mathbb{A}_k^{n+1} \setminus \{0\} \\ & \overset{\widetilde{\varphi}_0}{\nearrow} & \downarrow \pi \\ \mathbb{A}_k^n & \xrightarrow[\varphi_0]{} & \mathbb{P}_k^n \end{array}$$

where $\widetilde{\varphi}_0$ maps $(y_1, ..., y_n)$ to $(1, y_1, ..., y_n)$. See that $\varphi_0 = \pi \circ \widetilde{\varphi}_0$ is continuous.
On the other hand, we can define the inverse morphism

$$\varphi_0 : U_0 \to \mathbb{A}_k^n$$
$$[x_0 : \cdots : x_n] \mapsto \left[\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}\right]$$

hence $\mathbb{A}_k^n \simeq U_i$, we get an affine piece. □

Corollary 14.3. The Zariski topology of $\mathbb{P}^n$ is the topology induced by the open covering $\mathbb{P}^n = \cup_{i=0}^n U_i$, where $\varphi_i : U_i \to \mathbb{A}_k^n$, Zariski topology on U_i.

We always use the standard affine piece U_i, actually an arbitray linear form can give us an affine piece!

Remark 14.4. Given a linear form $L = \sum_{i=0}^n a_i x_i$, Without loss of generality, we may assume $a_0 \neq 0$. Consider the hypersuface cut by L, which means $H_L = V(L) \subseteq \mathbb{P}^n$ with

$$\varphi_L : \mathbb{A}_k^n \to \mathbb{P}^n \setminus H_L$$
$$(y_1, ..., y_n) \mapsto \left[\frac{1 - \sum_{i=1}^n a_i x_i}{a_0} : y_1 : \cdots : y_n\right]$$

which gives an isomorphism to $\mathbb{A}_k^n$!

Remark 14.5 (Basis of Zariski topology on $\mathbb{P}^n$)**.** Given F a homogeneous polynomial, consider $D(F)$, then open subsets of this form can form a basis of the topology, called $D(F)$ standard open subset.

14.1 §C. Structure Sheaf of Projective Space

Definition 14.6. The structure sheaf $\mathcal{O}_{\mathbb{P}^n}$ is the unique sheaf on $\mathbb{P}^n$ such that

$$\mathcal{O}_{\mathbb{P}^n}|_{U_i} \simeq \mathcal{O}_{\mathbb{A}_k^n} \quad 0 \leq i \leq n$$

Proposition 14.7. $\Gamma(D(F), \mathcal{O}_{\mathbb{P}^n}) = \left\{\frac{G}{F^m} \middle| G \text{ homogeneous with } \deg G = m \cdot \deg F\right\}$, see its sections are well-defined on $\mathbb{P}^n$.

Note that $D(F) \cap U_i = D(f_i) \subseteq \mathbb{A}_k^n \xrightarrow{\varphi_i} U_i$.

Remark 14.8. $\mathcal{O}_{\mathbb{P}^n}$ = functions are locally defined by two homogeneous polynomials of the same degree i.e. $\frac{F}{G}$ of same degree.

Theorem 14.9 (Main theorem!)**.** *The locally ringed space* $(\mathbb{P}^n, \mathcal{O}_{MP})$ *is a complete variety.*

Corollary 14.10. Closed subvarieties of $\mathbb{P}^n$ is also complete since closed subset of a complete space is complete.

It is a long proof.

Proof. (1) separateness:

□

Let's recall the basic properties of completeness.

Recall 14.11 (Basic properties of completeness)**.**

(1) Let $f : X \to Y$ be a morphism of varieties. If X is complete, then $f(X)$ is closed and again complete, closed map and image is complete(like image of compact is compact.).

(2) If X and Y are complete, then so does $X \times Y$.

(3) If X is complete and $Y \subseteq X$ is a closed subvariety, then Y is complete.

(4) Affine algebraic variety if complete $\iff$ its dimension is 0(hence finite points).

14.2 §D. Projective Varieties

Definition 14.12. A projective(resp. quasi-projective) variety is a closed subvariety(resp. variety) of $\mathbb{P}^n$.

Remark 14.13. Closed subvariety in $\mathbb{P}^n$ is also complete.

Basic facts 14.14. Let $X \subseteq \mathbb{P}^n$ be a projective variety, and $R = \oplus_{i=0}^{\infty} R_i = k[x_0, ..., x_n]$. Then

(1) Zariski topology on X.

$D(F) = \{x \in X | F(x) \neq 0\}$, $F \in R_i$ for some $i > 0$. Then it forms a basis for the Zariski topolpgy on X.

(2) Global sections on a complete irreducible/connected variety X are only constants.

Proof. Let $f \in \Gamma(X, \mathcal{O}_X)$, consider

$$f : X \to k = \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$$

then $f(X)$ is a closed subset of $\mathbb{P}^1_k$ since X is complete, and $f(X)$ = finite points by it is closed in $\mathbb{A}^1_k$, and we embed $\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ by $x \mapsto [1 : x]$, next, using X is connected, we get $f(X)$ is a single point since in $\mathbb{P}^1_k$ a set of finite points is connected if and only if it is a singleton, or we can find two disjoint closed subsets to cover it. □

(3) Regular functions on $D(F)$.

$$\Gamma(D(F), \mathcal{O}_X) = \left\{ \frac{G}{F^m} \middle| \deg G = m \cdot \deg F,\ m \geq 0\ G \in k[x_0, ..., x_n] \text{ homogeneous} \right\} \Big/ \sim$$

where $\frac{G}{F^m} \sim \frac{G'}{F^{m'}}$ if $\frac{G}{F^m}\big|_{D(F)} = \frac{G'}{F^{m'}}\big|_{D(F)}$.

Recall 14.15. We have known that projective varieties are complete, but does any complete variety is projective? The answer is false, Nagata gave a counterexample. However, Chow gave his Chow's lemma.

Lemma 14.16 (Chow's lemma)**.** [Mumford The Red Book Chapter I. §10] Let X be a complete variety over an algebracially closed field. Then there exists a projective variety Y and a birational surjective morphism

$$\pi : Y \to X$$

hence it is not far from a complete variety to a projective variety.

For example of nonprojective complete variety, due to Hironaka, see [[Har77] Appendix B Example 3.4.1.].

Algebraic Geometry I is over!

Let's come to Algebraic Geometry II which concerns divisors, vector bundles and cohomoloy!

Part II

Algebraic Geometry II

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Chapter VII Vector Bundles on Varieties

References:

(1) [Har77] Chapter II §5. §6. §7.

(2) [Mus] Chapter g §11.6.

15.1 §A. Definition and Examples

Definition 15.1 (Vector bundle)**.** A **vector bundle** V on a variety X is a variety with a surjective morphism

$$p : V \to X$$

such that there exists an open covering $X = \cup_{i \in I} U_i$(you can take a finite subcover) satisfies

(1) there exists isomorphism of varieties

$$\varphi_i : p^{-1}(U_i) \simeq U_i \times k^n = U_i \times \mathbb{A}_k^n \quad \forall i$$

commuting with p, i.e.

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\varphi_i} & U_i \times k^n \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U_i & \end{array}$$

(2) $\forall i, j \in I$, the composition

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times k^n \to (U_i \cap U_j) \times k^n$$

is fiberwise k-linear, i.e. for any point $x \in U_i \cap U_j$, the restricted morphism

$$\varphi_i \circ \varphi_j^{-1} : \{x\} \times k^n \to \{x\} \times k^n$$

is an isomorphism of k-vector space(see it is an isomorphism since we can give it an inverse.). So, although φ_i is a morphism of varieties, we still write k^n rather $\mathbb{A}_k^n$, since we want to mention the structure of vector space.

Example 15.2.

(1) (Tautological line bundle on $\mathbb{P}^n$)

We define

$$\mathcal{O}_{\mathbb{P}^n}(-1) := \left\{[x_0 : \cdots : x_n; \lambda x_0, ..., \lambda x_n] \in \mathbb{P}^n \times \mathbb{A}_k^{n+1} \middle| \lambda \in k\right\}$$

since we can regard $\mathbb{P}^n$ as a set of one-dimensional subspaces of $\mathbb{A}_k^{n+1}$. Then the fiber of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $[l] \in \mathbb{P}^n$ is exactly the line l in k^{n+1}.

We claim that

$$p : \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$$

is a line bundle on $\mathbb{P}^n$, where p is the first projection. Now, Let's check the transition functions.

Let $U_i = D(x_i) \subseteq \mathbb{P}^n$, then

$$\varphi_i : p^{-1}(U_i) \to \mathbb{A}_k^n \times k$$

by

$$\left[x_0 : \cdots : x_n; \lambda\frac{x_0}{x_i}, ..., \lambda\frac{x_n}{x_i}\right] \mapsto \left[\frac{x_0}{x_i}, ..., \widehat{\frac{x_i}{x_i}}, ..., \frac{x_n}{x_i}; \lambda\right]$$

consider $\psi_{ij} = \varphi_j \circ \varphi_i^{-1}$, we want to know the transition function, which means how does it glue or what's the action on k.

$$\varphi_j \circ \varphi_i^{-1} : U_i \cap U_j \times k \to U_i \cap U_j \times k$$

Let's break the map in steps:

$$\varphi_i^{-1} : [x_0, \cdots, \widehat{x_i}, \cdots, x_n; k] \mapsto \left[x_0 : \cdots : 1 : \cdots : x_n; k\frac{x_0}{x_i}, ..., k, ..., k\frac{x_n}{x_i}\right]$$

see that

$$\left[x_0 : \cdots : 1 : \cdots : x_n; k\frac{x_0}{x_i}, ...k, ..., k\frac{x_n}{x_i}\right] = \left[\frac{x_0}{x_j} : \cdots : \frac{x_n}{x_j}; k\frac{x_j}{x_i} \cdot \frac{x_0}{x_j}, ..., k\frac{x_j}{x_i} \cdot \frac{x_n}{x_j}\right]$$

then

$$\varphi_j : \left[x_0 : \cdots : 1 : \cdots : x_n; k\frac{x_0}{x_i}, ..., k\frac{x_n}{x_i}\right] \mapsto \left[\frac{x_0}{x_j} : \cdots : \frac{x_n}{x_j}; k\frac{x_j}{x_i}\right]$$

hence

$$\psi_{ij} : U_i \cap U_j \mapsto \frac{x_j}{x_i}$$

the action on k.

Remark 15.3. In the affine part, we always use $\frac{x_i}{x_j}$ to get a well-defined formula.

(2) (Hyperplane bundle)

We define

$$\mathcal{O}_{\mathbb{P}^n}(1) = \text{ dual bundle of } \mathcal{O}_{\mathbb{P}^n}(-1)$$

which means we attach the one-dimensional vector space structure of dual map of the line l in k^n to $\mathbb{P}^n$.

Transition fucntion of $\mathcal{O}_{\mathbb{P}^n}(1)$ is

$$\psi_{ij}^* : D(x_1) \cap D(x_2) \to \mathrm{GL}_1(k) = k^\times$$

by

$$[x_0 : \cdots : x_n] \mapsto \frac{x_i}{x_j}$$

(3) (Tangent bundle of a nonsingular irreducible affine variety) Let $V \subseteq \mathbb{A}_k^n$ be an irreducible affine algebraic variety, and $I(V) = (F_1, ..., F_m)$. We define Zariski tangent bundle T_V^{Zar} as

$$T_V^{Zar} := \left\{ (x, v) \in V \times k^n \middle| \sum_{j=1}^n \frac{\partial F_i}{\partial x_j}(x) \cdot v_j = 0 \quad 1 \leq i \leq m \right\}$$

which means equipping each point its tangent space.

For any $x \in V$, we have

$$\begin{aligned} T_{V,x}^{Zar} = p^{-1}(x) &= \left\{ v \in k^n \middle| \sum_{j=1}^n \frac{\partial F_i}{\partial x_j}(x) \cdot v_j = 0 \quad 1 \leq i \leq m \right\} \\ &= T_x V - x \subseteq k^n = \mathbb{A}_k^n \end{aligned}$$

where

$$p : T_V^{Zar} \to V$$

is the first projective. Since V is irreducible and nonsingular, we have $\dim T_{V,x}^{Zar} = \dim V = r$.

Here are some interesting examples of vector bundles.

Example 15.4.

(1) **Trivial vector bundle**.

$V \times k^n$.

(2) Let V_1 and V_2 be vector bundles on X. Then we can construct new vector bundles using algebraic operations.

(2.a) **Direct sum**.

$V_1 \oplus V_2$ such that $(V_1 \oplus V_2)(x) \simeq V_1(x) \oplus V_2(x)$, where $V_i(x) = p^{-1}(x)$.

(2.b) **Tensor product**.

$V_1 \otimes V_2$ such that $(V_1 \otimes V_2)(x) \simeq V_1(x) \otimes V_2(x)$.

(2.c) **Exterior power**.

$i \geq 1$, $\bigwedge^i V_1$, such that $(\bigwedge^i V_1)(x) \simeq \bigwedge^i(V_1(x))$.

(2.d) **Symmetric power**.

$i \geq 1$, $\mathrm{Sym}^i V_1$ such that $\left(\mathrm{Sym}^i V_1\right)(x) \simeq \mathrm{Sym}^i(V_1(x))$.

(2.e) **Dual bundle**.

V_1^* such that $V_1^*(x) \simeq (V_1(x))^*$.

(2.f) **Determinant**.

$\det V_1$ such that $\det V_1(x) \simeq \bigwedge^r(V_1(x))$, where $r = \mathrm{rank}(V_1)$.

Definition 15.5 (Line bundle)**.** A vector bundle of rank 1 is called a **line bundle**.

Remark 15.6 (Transition functions)**.**

(1) We can regard

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times k^n \to (U_i \cap U_j) \times k^n$$

as a morphism, more explicitly

$$\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_n(k)$$

then a vector bundle of rank r over a variety is determined by the following data:

i. An open covering $X = \cup U_i$.

ii. A family of transition functions

$$\left\{\psi_{ij} : U_i \cap U_j \to \mathrm{GL}_n(k) \subseteq \mathbb{A}_k^{n^2}\right\}$$

such that

$$\psi_{jl} \circ \psi_{ij}|_{(U_i \cap U_j \cap U_l)} = \psi_{il}|_{(U_i \cap U_j \cap U_l)} \quad \forall i, j, l$$

(2) Every algebraic operation of vector bundle can be translated as an algebraic operation of transition functions.

i. $V \rightsquigarrow V^*$.

$(U_i, \psi_{ij}) \rightsquigarrow (U_i, (\psi_{ij}^{-1})^t)$, wher $(\psi_{ij}^{-1})^t$ is the transposition of the inverse matrix.

ii. $V \rightsquigarrow \det V$.

$(U_i, \psi_{ij}) \rightsquigarrow (U_i, \det \psi_{ij})$, where $\det \psi_{ij}$ is the determinant of matrix.

Given a point $x \in X$, the matrix $\left[\frac{\partial F_i}{\partial x_j}(x)\right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ has rank $n - r$ as V is non-singular and of dimension r. Without loss of generality, we may assume the upper $(n - r)$ block A has nonzero determinant, define $G = \det A$ and let $U = D(G) \subseteq V$Something left

Definition 15.7 (Homomorphism of vector bundles)**.** Let V_1 and V_2 be vector bundles on X. A homomorphism

$$f : V_1 \to V_2$$

is a morphism of varieties commuting with projections on X i.e.

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad f \quad} & V_2 \\ & {\scriptstyle pr_1} \searrow \quad \swarrow {\scriptstyle pr_2} & \\ & X & \end{array}$$

and $f(x) := f|_{V_1(x)} : V_1(x) \to V_2(x)$ is a k-linear map of rank independent of x, which means it is a constant rank.

Example 15.8. $X = \mathbb{A}_k^1$, $V_1 = \mathbb{A}_k^1 \times k$, and $V_2 = \mathbb{A}_k^1 \times k$, take

$$f : V_1 \to V_2$$

by

$$(x, v) \mapsto (x, xv)$$

is not a homomorphism of vector bundles since its map at 0 is a zero map of rank 0, but rank 1 at any other points.

Definition 15.9 (Pullback)**.** Let $f : X \to Y$ be a morphism of varieties. let V be a vector bundle on Y. Then the pullback(fiber product) f^*V is a vector bundle on X such that

$$f^*V(x) = V(f(x))$$

more precisely, if V is given by (U_i, ψ_{ij}) then f^*V is given by $(f^{-1}(U_i), \psi_{ij} \circ f)$

Caution 15.10. Pushout may <u>NOT</u> be a vector bundle.

15.2 §B. Picard Group

Definition 15.11 (Picard group)**.** Let X be a variety. The **Picard group** $\mathrm{Pic}(X)$ of X is defined as the set of line bundles over X modulo isomorphic equivalence, which means

$$\mathrm{Pic}(X) = \{\text{line bundles on } X\} / \sim$$

where $L \sim L'$ if $L \simeq L'$ as vector bundles with

(a) Zero element: trivial line bundle $X \times k$.

(b) Multiplicity: $L \cdot L' := L \otimes L'$.

(c) Inverse: $L^{-1} = L^*$, the dual bundle.

Remark 15.12.

(1) Without loss of generality, we can always assume L and L' are given by (U_i, ψ_{ij}) and (U_i, ψ'_{ij}). Then

$$L \otimes L' = (U_i, \psi_{ij} \circ \psi'_{ij})$$

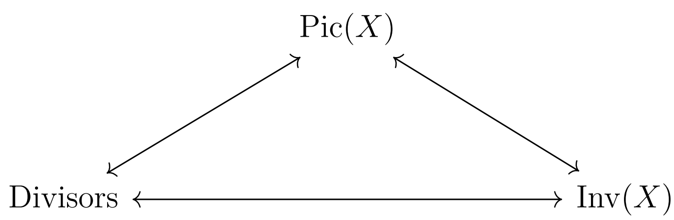
(2) $\mathrm{Pic}(X)$ is an abelian group.

Example 15.13 ($\mathrm{Pic}(\mathbb{P}^n)$)**.** $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^n}(1)$ denoted by

$$\mathcal{O}_{\mathbb{P}^n}(m) = \begin{cases} \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes m} & m > 0 \\ \mathcal{O}_{\mathbb{P}^n} & m = 0 \\ \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes(-m)} & m < 0 \end{cases}$$

transition functions for $\mathcal{O}_{\mathbb{P}^n}(m) : \psi_{ij} = (\frac{x_i}{x_j})^m$, $m \in \mathbb{Z}$.

In the remaining part of this chapter, we will study these sets of objects:



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16.1 §C. Weil Divisors and Cartier Divisors

In this subsection, we always assume X is an irreduvible variety.
(1) Weil Divisors

Definition 16.1 (Weil divisor)**.**

(1) A **prime divisor** on X is an irreducible codimension 1 closed subvariety of X.

(2) A **Weil divisor** D on X is a finite formal linear combination of prime divisors with coefficients in $\mathbb{Z}$, i.e.

$$D = n_1 D_1 + \cdots + n_r D_r$$

with $n_i \in \mathbb{Z}$ and D_i are prime divisors.

(3) The group $\mathrm{Div}(X)$ of Weil divisors on X is the free abelian group generated by prime divisors of X with

(a) $0 = 0$.

(b) $D + D' = \sum(n_i + n_i')D_i$ where $D = \sum n_i D_i$ and $D' = \sum n_i' D_i$.

(c) $-D = \sum(-n_i)D_i$.

(4) A Weil divisor D is called **effective** if all its coefficients are non-negative, in this case, we write $D \geq 0$.

(5) Given two Weil divisors D and D', we write $D \geq D'$ if $D - D' \geq 0$.

Recall 16.2. Let X be an irreducible normal variety, for any prime divisor D of X, there exists an affine open $U \subseteq X$ such that

(a) $U \cap D \neq \varnothing$.

(b) $\exists h \in \Gamma(U, \mathcal{O}_X)$ such that $I_U(D) = (h) \subseteq \Gamma(U, \mathcal{O}_X)$, where $I_U(D)$ is the ideal of $U \cap D$ in U.

Definition 16.3 (Principal divisor)**.** Let X be an irreducible normal variety. Given a non-zero rational $\phi \in K(X)$, we define a Weil divisor on X as

$$\operatorname{div}(\phi) := \sum_D \operatorname{ord}_D(\phi) \cdot D$$

which is the **principal divisor** associated to ϕ, D runs all prime divisors.

Remark 16.4.

(1) In the definition, the integer $\operatorname{ord}_D(\phi)$ is defined as following: given a divisor D, we choose an affine open U of X such that

(a) $U \cap D \neq \varnothing$.

(b) $I_U(D) = (h)$, $h \in \Gamma(U, \mathcal{O}_X)$.

write $\phi|_U = \frac{f}{g}$ where $f, g \in \Gamma(U, \mathcal{O}_X)$, then we define

$$\operatorname{ord}_D(\phi) = \operatorname{ord}_h(f) - \operatorname{ord}_h(g)$$

(2) $\operatorname{div}(\phi)$ is a finite sum.

In fact, there exists an affine open $U \subseteq X$ such that $\phi \in \Gamma(U, \mathcal{O}_X)$ and $\phi \neq 0$, hence there exists $U' \subseteq U$ an affine open such that $\phi|_{U'}$ nowhere vanishes. In particular, if $\operatorname{ord}_D(\phi) \neq 0$, then $D \subseteq X \setminus U'$, moreover number of D is finite, since we can decompose $X \setminus U'$ into finite disjoint union of irreducible components by Noether property, and D has codimension 1, hence only finite choice.

Definition 16.5 (Class group and linear equivalence)**.**

(1) The canonical map

$$\mathrm{div} : (K(X))^* \to \mathrm{Div}(X)$$

is a group homomorphism of abelian group.

(2) The **class group**: $\mathrm{Cl}(X)$ is the quotient $\mathrm{Div}(X)/\mathrm{Im}(\mathrm{div})$. For a Weil divisor D, we write $[D]$ for the class of D in $\mathrm{Cl}(X)$.

(3) Two Weil divisors are **linearly equivalent** if $[D'] = [D]$ i.e. there exists $\phi \in K(X)$ such that $D = D' + \mathrm{div}(\phi)$, write $D \sim D'$.

(2) Cartier Divisor

Definition 16.6. Let X be an irreducible normal variety. A Weil divisor D on X is called **Cartier** if D is **locally principal** i.e. there exists an open covering $X = \cup U_i$ and $\phi_i \in K(U_i)$ such that

$$\sum n_j(D_j \cap U_i) = D \cap U_i = \mathrm{div}(\phi_i) \quad \forall i \in I$$

where $D = \sum n_j D_j$.

Lemma 16.7 (Effective Cartier divisors)**.** Let X be an irreducible normal variety and $D = \sum n_i D_i$ be an effective Cartier diviosr(since it is a Weil divisor, so we have the sense of effective Cartier divisor) given by $X = \cup U_i$ and $\phi_i \in K(U_i)$. Then $\phi_i \in \Gamma(U, \mathcal{O}_X)$, namely regular.

Proof. Since D is effective, there exists $Z_i \subseteq U_i$ of codimension ≥ 2(Why?) such that $\phi_i \in \Gamma(U_i \setminus Z_i), \mathcal{O}_X$, more, U_i are normay by X is normal. By Hartogs Extension, we have

$$\Gamma(U_i, \mathcal{O}_X) \twoheadrightarrow \Gamma(U_i \setminus Z_i, \mathcal{O}_X)$$

□

Recall 16.8 (Serre's criterion)**.** Let A be a Noetherian ring.

(1) $R_k : A_{\mathfrak{p}}$ is a regular local ring for any prime ideal $\mathfrak{p}$ of $\mathrm{ht}(\mathfrak{p}) \leq k$.

(2) $S_k : \mathrm{depth}\, A_{\mathfrak{p}} \geq \inf\{k, \mathrm{ht}(\mathfrak{p})\}$ for any prime ideal $\mathfrak{p}$.

Then

(1) A is a reduced ring $\Longleftrightarrow R_0, S_1$ hold.

(2) A is a normal ring $\Longleftrightarrow R_1, S_2$ hold.

(3) A is Cohen-Macaulay $\Longleftrightarrow S_k$ hold for all k.

Example 16.9 (Weil divisors which are <u>NOT</u> Cartier). $X = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}_k^4 \Rightarrow X$ is irreducible and normal by Serre's criterion(if X is an irreducible normal variety, and Cohen-Macaulay, then it is normal, we take regular sequence $\{x_1, x_2, x_3 + x_4\}$).
Consider $D_1 = \{x_1 = x_3 = 0\} \cap X$ and $D_2 = \{x_2 = x_4 = 0\} \cap X$ both are isomorphic to $\mathbb{A}_k^2$. Then $D_1 \cap D_2 = \{0, 0, 0, 0\}$. Hence $\dim D_1 = \dim D_2 = 2$, $\dim D_1 \cap D_2 = 0$.
Then D_1 and D_2 are not Cartier. In fact, we may assume that D_1 is defined by $\phi \in \Gamma(U, \mathcal{O}_X)$ where $0 \in U$ a open neighborhood. Then $\phi|_{D_2} \neq 0$, however, by Krull's principal theorem

$$0 = \dim(D_1 \cap D_2) = \dim(V(\phi|_{D_2})) = \dim D_2 - 1 = 1$$

it is a contradiction.

Example 16.10. Principal divisor is Cartier.

Definition 16.11. Let X be an irreducible normal variety.

(1) X is called **factorial** if all Weil divisors on X are Cartier.

(2) X is called **$\mathbb{Q}$-factorial** if for any Weil divisor D, there exists $m \in \mathbb{N}$ depending on D such that mD is Cartier.

Example 16.12.

1. Using the same argument $X = V(x_1x_2 - x_3x_4) \subseteq \mathbb{A}_k^4$ is <u>NOT</u> $\mathbb{Q}$-factorial, since its dimension of $D_1 \cap D_2$ is wrong.

2. $X = V(x_1x_2 - x_3^2) \subseteq \mathbb{A}_k^3$, consider $D = \{x_1 = x_3 = 0\} \subseteq X$. Then D is not Cartier, however $2D$ is Cartier, which is defined as $\operatorname{div}(x_1)$. Let's compute it in details:

Proposition 16.13 ([Har77] Chapter II Proposition 6.11.). Let X be an irreducible normal variety, if $\mathcal{O}_{X,x}$ is a UFD for any $x \in X$, then X is factorial. In particular, if X is non-singular, then X is factorial(since regular local ring is UFD).

Notation 16.14. The subgroup $\mathrm{CaCl}(X)$ of $\mathrm{Cl}(X)$ which generated by Cartier divisors.

Definition 16.15 (Pullback of Cartier divisors). Let $f : Y \to X$ be a morphism of irreducible normal varieties, let D be a Cartier divisor on X given by

(i) $X = \cup U_i$.

(ii) $\phi_i \subseteq K(U_i) = K(X)$.

then the **pullback** f^*D is a Cartier divisor given by

(i) $Y = \cup f^{-1}(U_i)$.

(ii) $\psi_i = \varphi_i \circ f \in K(f^{-1}(U_i)) = K(Y)$.

Remark 16.16. In general, we can <u>NOT</u> define the pullback of a Weil divisor.

17 Lecture 17.

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17.1 §D. From Cartier Divisor to Line Bundles

Let X be an irreducible normal variety, and D be a Cartier divisor on X given by $X = \cup U_i$ and $\phi_i \in K(U_i) = K(X)$.
See that $\operatorname{div}(\phi_1)|_{U_1\cap U_2} = D|_{U_1\cap U_2} = \operatorname{div}(\phi_2)|_{U_1\cap U_2}$, where we can view $D|_{U_1\cap U_2}$ as zeros and poles of ϕ_1 and ϕ_2 in $U_1 \cap U_2$, hence $\frac{\phi_1}{\phi_2}|_{U_1\cap U_2}$ has no zero and pole, which means $\frac{\phi_1}{\phi_2} : U_1 \cap U_2 \to k^*$.
Define $\psi_{ij} = \frac{\phi_j}{\phi_i} \in K(U_i \cap U_j)\ \forall i, j$. Then $\psi_{ij} : U_i \cap U_j \to k^*$.
Hence, $\psi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$ and ψ_{ij} nowhere vanishes.

Definition 17.1 (Line bundle associated to D)**.** The line bundle L_D on X associated to D is the line bundle given by

(i) $X = \cup U_i$.

(ii) $\psi_{ij} = \frac{\phi_j}{\phi_i} : U_i \cap U_j \to k^* = \mathrm{GL}_1(k)$.

Picture.

Remark 17.2. Easy to see that

$$\psi_{il}|_{U_i\cap U_j\cap U_l} = \psi_{jl} \cdot \psi_{ij}|_{U_i\cap U_j\cap U_l}.$$

Definition 17.3 (Rational and global sections of line bundles)**.** Let L be a line bundle on an irreducible normal variety X given by the following data

(i) $X = \cup U_i$ an open covering with $L|_{U_i} \simeq U_i \times k$.

(ii) $\psi_{ij} : U_i \cap U_j \to k^*\ \forall i, j$.

(1) A **rational section** of L is given by the following data:

  (1.a) $\{s_i\}$ with $s_i \in K(U_i) = K(X)$.

  (1.b) $s_j = \psi_{ij} \cdot s_i$ in $K(U_i \cap U_j) = K(X)\ \forall\ i, j$.

  For example: Picture

(2) A **global section** of L is a rational section $\{s_i\}$ such that $s_i \in \Gamma(U_i, \mathcal{O}_X)$.

  Picture

Remark 17.4.

(1) $\{\text{global sections of } L\} \overset{1:1}{\longleftrightarrow} \{s : X \to L \text{ morphism such that } \pi \circ s = \mathrm{Id}_X,\ \pi : L \to X\}$

(2) Let D be a Cartier divisor given by $X = \cup U_i$ and $D \cap U_i = \operatorname{div} \phi_i$ with $\phi_i \in K(U_i)$, then $\{\phi_i\}$ is a rational divisor, just use natural transition function. In particular, if D is effective, then $\phi_i \in \Gamma(U_i, \mathcal{O}_X)$ and $\{\phi_i\}$ is a global section of L_D.

Lemma 17.5. Let D be a principal divisor, then L_D is isomorphisc to the trivial line bundle.

Proof.

$$\begin{aligned} D \text{ is principal } &\iff D = \operatorname{div}(\phi) \text{ some } \phi \in K(X) \\ &\iff L_D \text{ is given by } X \times k. \end{aligned}$$

□

By the lemma, we get a homomorphism of abelian groups

$$\mathrm{CaCl}(X) \xrightarrow{L} \mathrm{Pic}(X)$$

by

$$[D] \mapsto [L_D]$$

and

$$[D + D'] \mapsto [L_D \otimes L_{D'}]$$

see it is well-defined by if D' is principal then $L_{D'}$ is trivial which is the identity element in $\mathrm{Pic}(X)$.

Lemma 17.6. The homomorphism L is injective.

Proof. Let D be a Cartier divisor with

(i) $X = \cup U_i$.

(ii) $D|_{U_i} = \operatorname{div}(\phi_i)$ $\phi_i \in K(U_i)$

such that $L_D \simeq X \times k$ as vector bundles.

Take

$$s : X \to X \times k$$

by

$$x \mapsto (x, 1)$$

We can have a diagram:

$$U_i \times k \xrightarrow{\simeq} L_D|_{U_i} \xrightarrow{\varphi_i \simeq} U_i \times k, \qquad U_i \times k \xrightarrow{\widetilde{\varphi_i} \simeq} U_i \times k$$

Let's define

$$s_i : U_i \to k$$

by

$$x \mapsto \widetilde{\varphi}_i^{-1}(x, 1) = (x, s_i(x)) \text{ see that } s_i(x) \neq 0$$

Note that $\{s_i\}$ is a global section of $L_D \to X$, hence

$$s_j|_{U_i \cap U_j} = \psi_{ij} \cdot s_i|_{U_i \cap U_j} = \frac{\phi_j}{\phi_i} \cdot s_i|_{U_i \cap U_j}$$

hence

$$\left.\frac{s_j}{\phi_j}\right|_{U_i \cap U_j} = \frac{s_i}{\phi_i} \text{ in } K(U_i \cap U_j) = K(X)$$

Define $\phi = \frac{\phi_i}{\phi_j} \in K(X)$, see that $\operatorname{div}(\phi)|_{U_i} = \operatorname{div}(\phi_i)$ on U_i, hence $D = \operatorname{div}(\phi)$. □

Definition 17.7 (Cartier divisor defined by rational sections)**.** Let $s = \{s_i\}$ be a rational section of a line bundle L, then we can define a Cartier divisor $\operatorname{div}(s)$ as following

$$\operatorname{div}(s) := \sum_{D \text{ prime}} \operatorname{ord}_D(s) \cdot D$$

where $\operatorname{ord}_D(s)$ is defined as $\operatorname{ord}_D(s_i)$ for $U_i \cap D \neq \varnothing$.

Remark 17.8. If $D \cap U_i \neq \varnothing$ and $D \cap U_j \neq \varnothing$, then $D \cap U_i \cap U_j \neq \varnothing$ since D is irreducible. We have

$$\operatorname{ord}_D(s_j) = \operatorname{ord}_D(s_i \cdot \psi_{ij}) = \operatorname{ord}_D(s_i)$$

hence $\operatorname{ord}_D(s)$ is well-defined with respect to i.

Lemma 17.9. $L : \operatorname{CaCl}(X) \to \operatorname{Pic}(X)$ is surjective.

Proof. Let $L \in \operatorname{Pic}(X)$ be a line bundle given by

(i) $X = \cup U_i$.

(ii) $\psi_{ij} : U_i \cap U_j \to k^*$.

Define a rational section $\{s_i\} \in L_D$ as following:

(1) $s_1 : U_1 \to k^*$ by $x \mapsto 1$.

(2) $s_i = \psi_{1i} \cdot s_1 \in K(U_1 \cap U_i) = K(U_i) = K(X)$ is well-defined since $\psi_{jl} \cdot \psi_{ij} = \psi_{1l}$.

Let $D = \operatorname{div}(s)$ be the Cartier divisor associated to $\{s_i\}$. Then L_D is

(1) $X = \cup U_i$.

(2) $\psi'_{ij} = \frac{s_i}{s_l} = \frac{\psi_{1j} \cdot s_1}{\psi_{1i} \cdot s_1} = \psi_{ij}$.

Hence $L_D \simeq L$. □

Remark 17.10.

(1) Recall that $\Gamma(X, \mathcal{O}_X) = k$ if X is irreducible and complete.

(2) We regard a line bundle L_D as a data:

$$L_D = \bigsqcup_{x \in X} L_x$$

where L_x is a one-dimensional k-vector space.

(3) A rational section/global section s of D is a 'rational'/ 'regular' map

$$X \xrightarrow{s} \bigsqcup_{x \in X} L_x$$

by

$$x \mapsto s(x) \in L_x \text{ or } \infty \text{ if } x \text{ is a pole.}$$

(4) A Cartier divisor can be regarded as zeros minus poles of a rational map

$$s : X \to \bigsqcup_{x \in X} L_x$$

(5) $\mathrm{CaCl}(X) \simeq \mathrm{Pic}(X)$ by

$$[D] \mapsto [L_D]$$

the other direction

$$[\mathrm{Div(s)}] \leftarrow [L]$$

where s is a rational section of L.

17.2 §E. Sheaf of Sections of Vector Bundles

(1) Sheaf of sections

Definition 17.11. Let $\pi : V \to X$ be a vector bundle. $\mathscr{V}$, the **sheaf of sections of** V, is the sheaf for any open subset $U \subseteq X$,

$$\Gamma(U, \mathcal{O}_{\mathscr{V}}) = \{s : U \to V \text{ morphism} | \pi \circ s = \mathrm{Id}_U\}$$

Picture

Remark 17.12. If V is given by

(i) $X = \cup U_i$.

(ii) $\psi : U_i \cap U_j \to \mathrm{GL}_r(k)$.

Then $\mathscr{V}|_{U_i} \simeq$ sheaf of sections of $U_i \times k^r \simeq \mathcal{O}_{U_i}^{\oplus r}$.

(2) Sheaf of $\mathcal{O}_X$-modules

Definition 17.13. Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$-modules** is a sheaf of abelian groups $\mathscr{F}$ such that for any open subset $V \subseteq X$, we have a $\Gamma(U, \mathcal{O}_X)$-module structure on $\mathscr{F}(U)$ and these structure are compatible with restriction maps: for any open $V \subseteq U$, we have

$$(a \cdot s)|_V = a|_V \cdot s|_V \quad \forall a \in \Gamma(U, \mathcal{O}_X) \text{ and } s \in \Gamma(U, \mathscr{F})$$

Example 17.14. The sheaf of sections $\mathscr{V}$ of a vector bundle over X is a $\mathcal{O}_X$-module.

$$\begin{aligned} f \cdot s : U &\to \mathscr{V} \\ s &\mapsto f(x)s(x) \end{aligned}$$

since locally, the sheaf of sections is isomorphic to the free sheaf $\mathcal{O}_U^{\oplus r}$, hence $f \cdot s \in \Gamma(U, \mathcal{O}_{\mathscr{V}})$.

Definition 17.15. Let $(X, \mathcal{O}_X)$ be a ringed space.

(1) Let $\mathscr{F}, \mathscr{G}$ be two sheaves of $\mathcal{O}_X$-modules. A morphism

$$\varphi : \mathscr{F} \to \mathscr{G}$$

of sheaves of $\mathcal{O}_X$-modules is a morphism of sheaves such that for any open subset $U \subseteq X$

$$\varphi_U : \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{G})$$

is a morphism of $\mathcal{O}_X(U)$-modules.

(2) A sheaf $\mathscr{F}$ of $\mathcal{O}_X$-module is **locally free** if there exists an open covering $X = \cup U_i$ such that $\mathscr{F}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ isomorphic as sheaves of $\mathcal{O}_{U_i}$-modules.

Example 17.16. If $\mathscr{V}$ is the sheaf of sections of a vector bundle, then it is locally free.

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Remark 18.1.

(1) E = a vector bundle of rank r, given by $\begin{cases} E = \cup U_i. \\ \psi_{ij} : U_i \cap U_j \to \mathrm{GL}_r(k). \end{cases}$

E^* = the dual bundle of E given by $\begin{cases} E^* = \cup U_i. \\ \psi^*_{ij} = (\psi_{ij}^{-1})^t : U_i \cap U_j \to \mathrm{GL}_r(k). \end{cases}$

check that for $A \in \mathrm{GL}(V) \rightsquigarrow (AL)(v) := L(A^{-1}v)$, where V is a vector space, $v \in V$, $L \in V^*$, a linear form.

(2) In the definition of pullback of a Cartier divisor, $f : X \to Y$ a morphism between irreducible normal varieties, D a Cartier on Y. The f^*D is well-defined if $f(X) \nsubseteq \mathrm{Supp}(D) = \cup D_i$, or, $\phi_i \circ f : f^{-1}(U_i) \to k = 0$, hence

$$f^*D = \begin{cases} X = \cup(f^{-1}(U_i)). \\ \varphi_i = \phi_i \circ f = 0. \end{cases}$$

which is not a Cartier divisor.

(3) Let X be an irreducible normal variety. $D \subseteq X$ is a prime divisor, $\phi \in K(X)^*$. $\phi = \frac{f}{g}$ where $f, g \in \Gamma(U, \mathcal{O}_X)$, $I_U(D) = \langle h \rangle$.

$$\mathrm{ord}_h(\phi) = \max \{m \in \mathbb{Z}_{\geq 0} | \text{such that } f \in \langle h^m \rangle\}$$

Here we use the Krull's intersection theorem to guarantee that $\mathrm{ord}_h(f) < \infty$.

Theorem 18.2 (Krull's intersection theorem)**.** *[[AM94] Cor 10.18.]* $\Gamma(U, \mathcal{O}_U) \rightsquigarrow A$ *is Noetherian,* $(1) \neq \alpha \subseteq A$ *an ideal, then* $\bigcap_{n \geq 0} \alpha^n = 0$*, hence* $\mathrm{ord}_h(f) < \infty$.

18.1 §F. From Cartier Divisors to Invertible Sheaves

Definition 18.3. $\mathrm{Inv}(X) := \{\text{invertible sheaves on } X\} / \sim$, where $L \sim L'$ if they are isomorphic as $\mathcal{O}_X$-modules.

Remark 18.4. $\mathrm{Inv}(X)$ is an abelian group.

(1) $0 = \mathcal{O}_X$.

(2) $L^{-1} = L^*$ the sheaf $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(L, \mathcal{O}_U)$.

(3) $L \cdot L' := L \otimes L'$, where $L \otimes L'$ is the sheaf associated to the presheaf $U \mapsto L(U) \otimes_{\Gamma(U, \mathcal{O}_X)} L'(U)$.

Definition 18.5. Let X be an irreducible normal variety and let D be a Weil divisor on X, we define a sheaf of $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ on X as

$$U \to \Gamma(U, \mathcal{O}_X(D)) := \{\phi \in K(U)^* = K(X)^* | \operatorname{div}(\phi)|_U + D|_U \geq 0\} \cup \{0\}.$$

Remark 18.6.

(1) $\Gamma(U, \mathcal{O}_X(D))$: rational functions on U with restriction on zeros and poles.

$$\text{zeros} \geq D^-, \text{ poles} \leq D^+, \text{ where } D = D^+ - D^-$$

(2) In general, we want to find rational functions on X with 'less' poles and 'enough' zeros, i.e. find $\phi \in K(X)^*$ such that $\text{Zeros}(\phi) \geq D^-$ and $\text{Poles}(\phi) \leq D^+$.

Proposition 18.7. Let X be an irreducible normal variety, D, D' are two Weil divisors on X. Then $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ as $\mathcal{O}_X$-modules $\iff D \sim D'$.

Proof. $\Leftarrow$: Assume that $D \sim D'$, hence there exists a $\phi \in K(X)^*$ such that $D = D' + \text{div}(\phi)$. Define

$$\Phi_U : \Gamma(U, \mathcal{O}_X(D)) \to \Gamma(U, \mathcal{O}_X(D'))$$

by

$$s \mapsto s \cdot \phi$$

$s \cdot \phi$ is well-defined by they are rational functions.

$$\begin{aligned} s \in \Gamma(U, \mathcal{O}_X(D)) &\iff \text{div}(s)|_U + D|_U \geq 0 \\ &\iff \text{div}(s)|_U + D'|_U + \text{div}(\phi)|_U \geq 0 \\ &\iff \text{div}(s \cdot \phi)|_U + D'|_U \geq 0 \\ &\iff s \cdot \phi \in \Gamma(U, \mathcal{O}_X(D')) \end{aligned}$$

Similiarly, we define

$$\Phi^{-1} : \Gamma(U, \mathcal{O}_X(D')) \to \Gamma(U, \mathcal{O}_X(D))$$

by

$$s \mapsto \frac{s}{\phi}$$

check that they are inverse to each other, hence we get the isomorphism.

$\Rightarrow$: Assume that $\varphi : \mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ as sheaf of $\mathcal{O}_X$-modules. Since X is an irreducible normal variety, we can remove a closed subset of codimension ≥ 2 such that $\widetilde{X}$ is non-singular. Without loss of generality, we may assume X is non-singular. Then D and D' are Cartier. Hence we get an open covering $X = \cup U_i$ and ϕ, ϕ' such that $D|_{U_i} = \text{div}(\phi)$ and $D'|_{U_i} = \text{div}(\phi_i')$ for any i. Let

$$\varphi_i := \varphi|_{U_i} : \Gamma(U_i, \mathcal{O}_X(D)) \to \Gamma(U_i, \mathcal{O}_X(D'))$$

Claim 1. For any i, $\exists! h_i \in K(U_i)^* = K(X)^*$ such that $\varphi_i(\cdot) = \cdot h_i$.

Proof. Note that $\frac{1}{\phi_i} \in \Gamma(U_i, \mathcal{O}_X(D))$, define $h_i = \phi_i \cdot \varphi(\frac{1}{\phi_i}) \in K(U_i)^*$. $\forall s \in \Gamma(U_i, \mathcal{O}_X(D))$, we have $\mathrm{div}(s)|_{U_i} + D|_{U_i} \geq 0$, hence $s \cdot \phi_i \in \Gamma(U_i, \mathcal{O}_X)$. In particular, we get

$$\varphi_i(s) = \varphi_i\left(s \cdot \phi_i \cdot \frac{1}{\phi_i}\right) = s \cdot \varphi_i \cdot \varphi\left(\frac{1}{\phi_i}\right) = s \cdot h_i \in \Gamma(U_i, \mathcal{O}_X(D'))$$

□

Claim 2. $\forall i, j$, $h_i = h_j \in K(X)^*$, we denote it by h.

Proof. Assume X is irreducible, hence $U_i \cap U_j \neq \varnothing$, and $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$ □

Claim 3. $D = D' + \mathrm{div}(h)$.

□

Corollary 18.8. Let X be an irreducible normal variety, D is a Weil divisor on X. Then D is Cartier $\iff \mathcal{O}_X(D)$ is an invertible sheaf.

Proof. $\Rightarrow$: Assume that D is Cartier, then $D = \begin{cases} X = \cup U_i \\ D|_{U_i} = \mathrm{div}(\phi_i)|_{U_i} \quad \phi_i \in K(X) \end{cases}$.

See that

$$\mathcal{O}_X(D)|_{U_i} \simeq \frac{1}{\phi_i}\mathcal{O}_{U_i} \simeq \mathcal{O}_{U_i}$$

$\Leftarrow$: Assume $\mathcal{O}_X(D)$ is invertible, hence there exists an open covering $\cup U_i$ such that $\mathcal{O}_X(D)|_{U_i} \simeq \mathcal{O}_{U_i}$, by the proposition above, we get $D|_{U_i} \sim 0$, so D is Cartier. □

18.2 §G. Summary

In diagram: Left.

Let X be an irreducible normal variety.

(1) $\mathrm{Pic}(X) = \{\text{line bundles on } X\}/\sim$, where $\sim$ means isomorphism of line bundles.

(2) $\mathrm{CaCl}(X) = \{\text{Cartier divisors on } X\}/\sim$, where $\sim$ means linearly equivalence.

(3) $\mathrm{Inv}(X) = \{\text{invertible sheaves on } X\}/\sim$, where $\sim$ means isomorphism of invertible sheaves.

(4) The diagr[AM94] is commutative of homomorphisms of abelian groups.

18.3 §H. Global Sections on Linear Systems

Let X be an irreducible normal variety.

D a Cartier divisor on X given by $\begin{cases} X = \cup U_i \\ D|_{U_i} = \text{div}(\phi_i),\ \phi_i \in K(U_i)^* = K(X)^* \end{cases}$.

$L_D =$ line bundle associated to D.

$\mathcal{O}_X(D) =$ invertible sheaf associated to D.

Remark 18.9 (From global sections of L_D to global sections of $\mathcal{O}_X(D)$)**.** Consider

Definition 18.10 (Complete linear system)**.** The **complete linear system** associated to D is the set

$$|D| := \{D' \in \text{Div}(X) | 0 \leq D' \text{ and } D' \sim D\}$$

Convention 18.11. If V is a vector space, we write $\mathbb{P}(V)$ for $\mathbb{P}(V \setminus \{0\})$, the projective space of V.

Definition 18.12. We define

$$\Phi : \mathbb{P}(\Gamma(X, L_D)) \to |D|$$

by

$$\bar{s} \mapsto \text{div}(s)$$

which is well-defined since $\text{div}(s) = \text{div}(\lambda s)$ for $\lambda \in k^*$.

See that

$$\begin{aligned} s = \{s_i\}_{i \in I} &\Rightarrow \text{div}(s)|_{U_i} = \text{div}(s_i) = \text{div}(\phi) + D|_{U_i} \geq 0 \\ &\Rightarrow \text{div}(s)|_{U_i} = D'|_{U_i} \geq 0 \text{ where } D' = \text{div}(\phi) + D \\ &\Rightarrow D' \geq 0 \\ &\Rightarrow D' \in |D| \end{aligned}$$

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Proposition 19.1.

(1) Φ is surjective.

(2) If X is projective, then Φ is injective.

(3) If $0 \neq s_1 \in \Gamma(X, L_{D_1})$ and $0 \neq s_2 \in \Gamma(X, L_{D_2})$, then $s_1 \otimes s_2 \in \Gamma(X, L_{D_1} \otimes L_{D_2})$, with $\text{div}(s_1 \otimes s_2) = \text{div}(s_1) + \text{div}(s_2) \in |D_1 + D_2|$.

Proof.

(1) Given $0 \leq D' \in |D|$, then $\exists 0 \neq \phi \in K(X)$ such that $D' = D + \mathrm{div}(\phi)$, hence $\phi \in \Gamma(X, \mathcal{O}_X(D))$, which means $\{\phi \cdot \phi_i\}$ defines a global section of L_D, where $\begin{cases} X = \cup U_i \\ D|_{U_i} = \mathrm{div}(\phi_i)|_{U_i} & \forall i \in I \end{cases}$

Note that $\phi \cdot \phi_j = \frac{\phi_j}{\phi_i}(\phi \cdot \phi_i)$.

(2) Assume X is projective, then $\Gamma(X, \mathcal{O}_X) = k$. If $\mathrm{div}(s_1) = \mathrm{div}(s_2)$, then $0 \neq \frac{s_1}{s_2} \in K(X)$ is regular, hence $[s_1] = [s_2]$ in $\mathbb{P}(\Gamma(X, L_D))$(affine case is false, see in $D(f)$, $\frac{1}{f}$ nowhere vanishes but not a constant).

(3) Easy by definition.

□

Remark 19.2. (1)+(2) means that for an irreducible normal projective variety X and a Cartier divisor D on X, the study of $|D|$ is equivalent to the study of the group of global sections of L_D, i.e. $\Gamma(X, L_D)$.

19.1 §I. Ample and Very Ample Line Bundles

(1) Global section of $\mathcal{O}_{\mathbb{P}^n}(m)$.

Recall 19.3. $[x_0 : \cdots : x_n]$ on $\mathbb{P}^n$, $s = k[x_0, ..., x_n]$ and $U_i = D(x_i) = \{x_i \neq 0\} \subseteq \mathbb{P}^n$. The line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$ is given by the data

(i) $X = \cup U_i$

(ii) $\psi_{ij} : U_i \cap U_j \to k^* \quad [x_0 : \cdots : x_n] \mapsto \frac{x_i^m}{x_j^m}$

Proposition 19.4.

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \begin{cases} S_m & m \geq 0 \\ 0 & m < 0 \end{cases}$$

Lemma 19.5. Let $0 \neq s \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)$ be a global section given by $\begin{cases} \mathbb{P}^n = \cup U_i \\ s_i = \frac{F_i}{x_i^d} & F_i \in S_d \\ s_j|_{U_i \cap U_j} = \frac{x_i^m}{x_j^m} \cdot s_i|_{U_i \cap U_j} \end{cases}$

(2) Base Locus of Linear System Let X be an irreducible normal variety, D a Cartier divisor on X, and $\Phi : \mathbb{P}\left(\Gamma(X, L_D)\right) \to |D|$.

Definition 19.6 (Linear system)**.** A **linear system** associated to D is the image of projective linear k-subspace of $\mathbb{P}\left(\Gamma(X, L_D)\right)$.

Definition 19.7 (Base locus of global sections of line bundle)**.** Let X be a variety, L a line bundle over X, $W \subseteq \Gamma(X, L)$ a liner k-subspace. Then the **base locus** of W is defined as

$$\mathrm{Bs}(W) := \{x \in X | s(x) = 0 \quad \forall s \in W\}$$

We say that W is **base point free** if $\mathrm{Bs}(W) = \varnothing$.

Remark 19.8. If $\dim W < \infty$ with $s_1, ..., s_r$ a basis, then $\mathrm{Bs}(W) = V(s_1) \cap \cdots \cap V(s_r)$, hence closed.

Definition 19.9 (Base locus of linear system)**.** Let X be an irreducible normal variety, D is a Cartier divisor over X, $W \subseteq |D|$ a linear system. The **base locus** of W is

$$\mathrm{Bs}(W) := \{x \in X | x \in \mathrm{Supp}(D') \quad \forall D' \in W\}$$

All linearly equivalent divisors must pass through x, hence you can't move divisors to avoid x. We say W is **base point free** if $\mathrm{Bs}(W) = \varnothing$.

Remark 19.10. Let $W \subseteq |D|$ be a linear system corresponding to linear k-subspace $W' \subseteq \Gamma(X, L_D)$. Then $\mathrm{Bs}(W) = \mathrm{Bs}(W')$

$$\Gamma(X, L_D) \to |D|$$

by

$$s \mapsto \mathrm{div}(s)$$

(3) Morphism to projective space defined by linear systems

Let L be a line bundle over a variety X. Take $W \subseteq \Gamma(X, L)$ a base point free finite dimensional linear k-subspace, we want to define a morphism $X \to \mathbb{P}(W^\vee)$ using

(a) First definition using a basis of W.

Let $s_0, ..., s_N$ be a basis of W over k. Then we have $W^\vee \simeq k^{N+1}$ using the dual basis $\{s_i^\vee\}$. Define

$$\Phi_W : X \to \mathbb{P}^N$$

by

$$x \mapsto [s_0(x) : \cdots : s_N(x)]$$

need to check Φ_W is well-defined.

Lemma 19.11. Φ_W is well-defined.

Remark 19.12. Since W is base point free, for any $x \in X$, there exists some i such that $s_i(x) \neq 0$, hence Φ_W is a morphism.

(b) Second definition without using a basis of W.

For $x \in X$, we define $W_x := \{s \in W | s(x) = 0\}$, and a valuation map

$$\nu_x : W \to k$$

by

$$s \mapsto s(x)$$

$\mathrm{Bs}(W) = \varnothing \Rightarrow \nu_x$ is a surjective k-linear map, hence $W_x \subseteq W$ is a linear k-subspcae of codimension 1.

Consider $(W/W_x)^\vee := W_x^\perp \subseteq W^\vee$, the annihilation of W_x, is an one-dimensional linear k-subspace, where $W_x^\perp = \{l \in W^\vee | \; l|_{W_x} = 0\}$.

Then, we define

$$\Phi_W : X \to \mathbb{P}(W^\vee)$$

by

$$x \mapsto W_x^\perp$$

Exercise 19.13. The two definition of Φ_W coincide.

(c) Morphisms defined by linear system.

Let X be an irreducible normal variety, D a Cartier divisor on X, $|W| \subseteq D$ a base point free finite dimensional k-linear subspace corresponding to $W' \subseteq \Gamma(X, L_D)$. Then we define

$$\Phi_{|W|} := \Phi_{W'} : X \to \mathbb{P}((W')^\vee)$$

(d) Rational map to projective space.

$W \subseteq \Gamma(X, L)$, a finite dimensional linear k-subspace, then

$$\Phi_{|W|} : X \dashrightarrow \mathbb{P}(W^\vee)$$

is the rational map given by

$$\Phi_{|W|} : X \setminus \mathrm{Bs}(W) \to \mathbb{P}(W^\vee)$$

by

$$W' = \mathrm{Im}(\Gamma(X, L)) \xrightarrow{res} \Gamma(X \setminus \mathrm{Bs}(W), L)$$

(4) Ample and very ample line bundles

Definition 19.14 (Very ample line bundle)**.** Let L be a line bundle on a variety X. We say that L is **very ample** if there exists $W \subseteq \Gamma(X, L)$, a base point free linear k-subspace and a subvariety $Z \subseteq \mathbb{P}(W^\vee)$ such that

$$\Phi_{|W|} : X \to \mathbb{P}(W^\vee)$$

induces an isomorphism

$$\Phi_{|W|} : X \simeq Z$$

Example 19.15.

(1) $\mathcal{O}_{\mathbb{P}^n}(1)$ is very ample.

$\Gamma(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n(1)}) = S_1$, choose $x_0 : \cdots : x_n$ as a basis, see

$$\mathbb{P}^n \to \mathbb{P}\left(W^\vee\right)$$

by

$$[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n]$$

where x_i on the right is the dual map.

(2) Choose a subvariety $X \subset \mathbb{P}^n$, then $\mathcal{O}_{\mathbb{P}^n}(1)|_X$ is very ample.

(3) $\mathcal{O}_{\mathbb{A}^n_k}$ is very ample.

take $1, Y_1, ..., Y_n \in \Gamma(\mathbb{A}^n_k, \mathcal{O}_{\mathbb{A}^n_k})$, $W = \text{spac}\{1, Y_1, ..., Y_n\}$, see

$$\mathbb{A}^n_k \to \mathbb{P}^n$$

by

$$(y_1, ..., y_n) \mapsto [1 : y_1 : \cdots : y_n]$$

Definition 19.16 (Ample line bundle)**.** Let L be a line bundle over a variety X, we say L is **ample** if there exists $m \in \mathbb{Z}^+$ such that $L^{\otimes n}$ is very ample.

Definition 19.17 (Ample and very ample Cartier divisors)**.** Let D be a Cartier on an irreducible normal variety X, we say D is ample(resp. very ample) if L_D is ample(resp. very ample).

Remark 19.18.

(1) D is ample $\iff mD$ is very ample, hence we can define ampleness for $\mathbb{Q}$-Cartier divisor, i.e. a $\mathbb{Q}$-Cartier divisor is ample if there exists m such that mD is a very ample Cartier divisor.

(2)

$$\{\text{ample line bundles on } X\} \overset{1:1}{\longleftrightarrow} \{\text{embedding } X \text{ into a projective space}\}$$

Example 19.19.

(1) (Linear projective space)

Let $L \subseteq \mathbb{P}^n$ be a projective linear subspace of dimension r. There exists $L_1, ..., L_{n-r}$ linearly independent linear functions such that $V(L) = (L_1, ..., L_{n-r})$(viewed as a linear system). Then the projective form is defined to be the rational map

$$\mathbb{P}^n \overset{\pi_L}{\dashrightarrow} \mathbb{P}_k^{n-r-1}$$

by

$$x \mapsto [L_1(x) : \cdots : L_{n-r}(x)]$$

π_L is regular on $\mathbb{P}^n \setminus L$.

(2) $F_0, ..., F_r \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, k-linearly independent, $W = \operatorname{span}\{F_0, ..., F_r\}$, see that

$$\Phi_W : \mathbb{P}^n \dashrightarrow \mathbb{P}(W^\vee)$$

by

$$[x_0 : \cdots : x_n] \mapsto [F_0(x) : \cdots : F_r(x)]$$

(3) (Veronese embedding)

$W_d^n = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ with $d \geq 1$. Then the d-th Vernoese embedding is the morphism

$$\mathbb{P}^n \xrightarrow{\nu_d} \mathbb{P}^N$$

by

$$[x_0 : \cdots : x_n] \mapsto [x_0^d : x_0^{d-1}x_1 : \cdots : x_n^d]$$

where $N = \dim(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) - 1 = \binom{n+d}{n} - 1$. Indeed, it is an embedding which means $\mathbb{P}^n \simeq \operatorname{Im}(\nu_d)$.

We can also view it as for $d \in \mathbb{Z}_{>0}$, $W_d^n = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = S_d$, then $\nu_d = \Phi_{|W_d^n|}$.

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Continue last lecture.

(4) (Plane conics)

Consider

$$\Phi_{|W_2^1|} : \mathbb{P}^1 \to \mathbb{P}^2$$

by

$$[x_0 : x_1] \mapsto [x_0^2 : x_0x_1 : x_1^2]$$

Let Z be the image $\Phi_{|W_2^1|}(\mathbb{P}^1)$. Then

$$I(Z) = \left(Y_1^2 - Y_0Y_2\right) \subseteq \mathbb{P}^3$$

(4) (Twisted cubic)

Consider

$$\Phi_{|W_2^1|} : \mathbb{P}^1 \to \mathbb{P}^3$$

by

$$[x_0 : x_1] \mapsto [x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3]$$

Let Z be the image $\Phi_{|W_2^1|}(\mathbb{P}^1) \subseteq \mathbb{P}^3$. Then Z is called the **twisted cubic** and

$$I(Z) = \left(Y_0Y_3 - Y_1Y_2, Y_1^2 - Y_0Y_2, Y_2^2 - Y_1Y_3\right)$$

(4) (Segre embedding)

$$\begin{array}{ccc} \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} & \xrightarrow{pr_2} & \mathbb{P}^{n_2} \\ \downarrow{\scriptstyle pr_1} & & \\ \mathbb{P}^{n_1} & & \end{array}$$

$\mathcal{O}_{\mathbb{P}^{n_1}} \times \mathcal{O}_{\mathbb{P}^{n_2}}(d_1, d_2) = pr_1 \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \otimes pr_2 \mathcal{O}_{\mathbb{P}^{n_2}}(d_2)$ with $d_1, d_2 \in \mathbb{Z}$. We claim that

$$W_{d_1,d_2^{n_1,n_2}} := \Gamma\left(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}, \mathcal{O}_{\mathbb{P}^{n_1}} \times \mathcal{O}_{\mathbb{P}^{n_2}}(d_1, d_2)\right) = \Gamma(\mathbb{P}^{n_1}, \mathcal{O}_{\mathbb{P}^{n_1}}(d_1)) \otimes \Gamma(\mathbb{P}^{n_2}, \mathcal{O}_{\mathbb{P}^{n_1}}(d_2))$$

Consider

$$\Phi_{|W_{1,1}^{n_1,n_2}|} : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \to \mathbb{P}^{(n_1+1)(n_2+1)-1}$$

by

$$[x_0 : \cdots : x_{n_1}] \times [y_0 : \cdots : y_{n_2}] \mapsto [x_0y_0 : x_0y_1 : \cdots : x_{n_1}y_{n_2}]$$

is called the Segre embedding of $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$.

Remark 20.1. Let L be a very ample line bundle, there exists $W \subseteq \Gamma(X, L)$ a base point free finite dimensional k-linear subspace with

$$\Phi_{|W|} : X \simeq Z \subseteq \mathbb{P}^N$$

where Z is a subvariety, then $L \simeq \Phi_{|W|} \mathcal{O}_{\mathbb{P}^N}(1)$

20.1 §J. Basic Properties of Ample Line Bundles

(1) Quasi-projective varities

Definition 20.2.

A variety X is **quasi-projective** $\iff$ there exists an isomorphism $\varphi : X \simeq Z \subseteq \mathbb{P}^n$ where Z is a subvarie

$\iff$ there exists an ample line bundle on X

Recall that Hironaka says there exists complete varieties that are not projective, hence there exists varieties having no ample line bundle.

(2) Twisting by Globally Generated Line Bundles

Definition 20.3. A line bundle L is globally generated if $\mathrm{Bs}(\Gamma(X, L)) = \varnothing$.

Remark 20.4. By Noetherian property of X, L is globally generated $\iff$ there exists an $W \subseteq \Gamma(X, L)$ finite dimensional such that $\mathrm{Bs}(W) = \varnothing$.

Proposition 20.5.

(1) L is very ample and L' is globally generated $\Rightarrow L \otimes L'$ is very ample.

(2) L and L' are globally generated $\Rightarrow L \otimes L'$ is globally generated.

(3) L is ample and L' is globally generated $\Rightarrow L \otimes L'$ is ample.

Fact 20.6. The graph of morphism is closed. More precisely, if $f : X \to Y$ is a morphism of varieties, then $\Gamma_f := \{x, f(x) \in X \times Y | x \in X\} \subseteq X \times Y$ is a closed subset. Because

$$\Gamma_f = \Phi^{-1}(\Delta_Y) \text{ where } \Delta_Y \text{ is closed.}$$

by

$$\Phi : X \times Y \xrightarrow{(f,\mathrm{Id})} Y \times Y \overset{\text{closed}}{\supseteq} \Delta_Y \text{ diagonal}$$
$$(x, y) \mapsto (f(x), y)$$

hence $\Gamma \subseteq X \times Y$ is closed.

Proof.

(1) Without loss of generality, we choose $W \subseteq \Gamma(X, L), W' \subseteq \Gamma(X, L')$ such that

  (a) $\Phi_{|W|} : X \simeq Z \subseteq \mathbb{P}^{N_1}$ an isomorphism.

  (b) $\Phi_{|W'|} : X \to \mathbb{P}^{N_2}$ a morphism.

  Then set $W = W \otimes W' \subseteq \Gamma(X, L \otimes L')$. See that

$$\begin{array}{ccc} \Phi : X & \xrightarrow{(\Phi_{|W|},\Phi_{|W'|})} & \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \\ & \underset{\Phi_{|W|}}{\overset{\simeq}{\searrow}} & \downarrow \\ & & Z \subseteq \mathbb{P}^{N_1} \end{array}$$

  Denote $Y = \Phi(X) \subseteq \mathbb{P}^N_1 \times \mathbb{P}^{N_2}$, is it locally closed? See $Y = \Phi(X) = \Gamma_f \overset{\text{closed}}{\subseteq} Z \times \mathbb{P}^{N_2}$, where $f := \Phi_{|W'|} \circ \Phi_W^{-1} : Z \to \mathbb{P}^{N_2}$. $Z \times \mathbb{P}^{N_2} \subseteq \mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$ is locally closed since $Z \subseteq \mathbb{P}^{N_1}$ is locally closed. Hence Γ_f is locally closed in $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$.

(2) $\forall x \in X$ take $s_1 \in \Gamma(X, L)$ and $s_2 \in \Gamma(X, L')$ such that $s_i(x) \neq 0$, then $s_1 \otimes s_2 \in \Gamma(X, L \otimes L')$ and $(s_1 \otimes s_2)(x) = s_1(x) \otimes s_2(x) \neq 0$.

(3)

$$\begin{aligned} L \text{ is ample } &\iff \exists m \in Z_{>0} \text{ such that } L^{\otimes m} \text{ is very ample} \\ &\overset{(1)}{\Rightarrow} L^{\otimes m} \otimes L' \text{ is very ample} \\ &\overset{(1)+(2)}{\Rightarrow} (L \otimes L')^{\otimes m} = L^{\otimes m} \otimes L^{\otimes m} = \left(L^{\otimes m} \otimes L'\right) \otimes L'^{\otimes (m-1)} \text{ is very ample} \\ &\Rightarrow L \otimes L' \text{ is ample} \end{aligned}$$

□

(3) Twisting an Ample Line Bundle

Lemma 20.7 (Extension of global sections, [Har77] Chapter II lem 5.14.)**.** Let $X \subseteq \mathbb{P}^N$ be a quasi-projective variety, $L = \mathcal{O}_{\mathbb{P}^n}(1)|_X$. Given a homogeneous polynomial $F \in \Gamma(X, L^{\otimes d})$ of degree $d > 0$, and a local section $s \in \Gamma(D(F), L)$. Then there exists $n \in \mathbb{Z}_{>0}$ such that $F^n \otimes s \in \Gamma(D(F), L^{\otimes nd} \otimes L')$ extends to a section $\widetilde{s} \in \Gamma(X, L^{\otimes nd} \otimes L')$ such that $\widetilde{s}|_{D(F)} = F^n \otimes s$.

Idea of proof. $s \in \Gamma(D(F), L')$. Thus we can view s as a rational section of L' with pole $\subseteq V(F)$, take n large enough such that $F^n s$ has no pole.

Proposition 20.8. L an ample line bundle, L' an arbitray line bundle. Then there exists $m \in \mathbb{Z}_{>0}$ such that $L^{\otimes m} \otimes L'$ is globally generated.

Proof.

Step 1. Reduce to the case where L is very ample.

Since there exists $m \in \mathbb{Z}_{>0}$ such that $L^{\otimes m}$ is very ample. If there exists $m' \in \mathbb{Z}_{>0}$ such that $(L^{\otimes m})^{\otimes m'} \otimes L'$ is globally generated, then $L^{\otimes mm'} \otimes L'$ is globally generated.

Step 2. Reduce to point.

More precisely, it is enough to show that for $\forall x \in X$ there exists $m \in \mathbb{Z}_{>0}$ such that there exists $s \in \Gamma(X, L^{\otimes m} \otimes L')$ with $s(x) \neq 0$.

Indeed, as L is very ample, for any $m \in \mathbb{Z}_{>0}$, we have $\mathrm{Bs}\left(\Gamma(X, L^{\otimes m_x + m} \otimes L')\right) \subseteq \mathrm{Bs}\left(\Gamma(X, L^{\otimes m_x} \otimes L')\right)$. See that $\Gamma(X, L^{\otimes m_x} \otimes L') \otimes \Gamma(X, L^{\otimes m}) \subseteq$left

□

Corollary 20.9. Let L be an ample line bundle over a variety X.

(1) $\exists n_0 > 0$ such that $L^{\otimes n}$ is globally generated for any $m > n_0$.

(2) $\exists m_0 > 0$ such that $L^{\otimes m}$ is very ample for any $m > m_0$.

Proof.

(1) Assuming the contracting that there exists $n_i \to +\infty$ such that $L^{\otimes n_i}$ is NOT globally generated. On the otherhand, there exists $m \in \mathbb{Z}_{>0}$ such that $L^{\otimes m}$ is very ample. After passing to a subsequence, we may assume $n_i = r_i m + c$ with $r_i \to +\infty$ $1 \le c \le m-1$. By the proposition above, there exists $m_c \in \mathbb{Z}_{>0}$ such that $L^{\otimes m \cdot m_c} \otimes L^{\otimes c}$ is globally generated $\Rightarrow$ if $r_1 > m_c$ then $L^{\otimes n_1} = L^{\otimes (r_1 - m_c)m} \otimes L^{m \cdot m_c + c}$ is globally generated.

(2) Take $m_0 = m + n_0$ where $L^{\otimes m}$ is very ample $\Rightarrow L^{\otimes m'} = \underbrace{L^{\otimes m}}_{\text{very ample}} \otimes \underbrace{L^{\otimes m' - m}}_{g.g.}$ for $m' \ge m + n_0 = m_0$, hence $L^{\otimes m}$ is very ample.

□

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Corollary 21.1. Let L be an ample line bundle and L' be an arbitary line bundle, then

(1) $\exists n_0$ such that $L^{\otimes n} \otimes L'$ is globally generated for any $n \ge n_0$.

(2) $\exists m_0$ such that $L^{\otimes m} \otimes L'$ is ample for any $m \ge m_0$.

(3) $\exists r_0$ such that $L^{\otimes r} \otimes L'$ is very ample for any $r \ge r_0$.

Proof. From Corollary 20.9, we know there exists $n' \in \mathbb{Z}_{>0}$ such that $L^{\otimes n'} \otimes L'$ is globally generated. Fined $n'' \in \mathbb{Z}_{>0}$ such that $L^{\otimes n}$ is very ample for $\forall n \ge n''$, hence $L^{\otimes n}$ is globally generated for $\forall n \ge n''$, and thus $L^{\otimes n} \otimes L'$ is globally generated for $\forall n \ge n' + n''$(by g.g.$\otimes$ g.g. is g.g.). □

Chapert VIII Quasi-coherent and Coherent Sheaves

References:

(1) [Har77] Chapter II, §5.

(2) Mumford Chapter III, §1, §2.

(3) [Mus] Chapter 8.

21.1 §A. Sheaves of Modules: Definition and Examples

(1) Recall: definition

Definition 21.2 (Sheaf of $\mathcal{O}_X$-modules)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of $\mathcal{O}_X$-modules** is a sheaf of abelian groups $\mathscr{F}$ such that for any open subset $V \subseteq X$, we have a $\Gamma(U.\mathcal{O}_X)$-module structure on $\mathscr{F}(U)$ and these structure are compatible with restriction maps: for any open $V \subseteq U$, we have

$$(a \cdot s)|_V = a|_V \cdot s|_V \quad \forall a \in \Gamma(U, \mathcal{O}_X) \text{ and } s \in \mathscr{F}(U, \mathcal{O}_X)$$

Remark 21.3.

(1) A sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ is **locally free** $\Longleftrightarrow$: $\exists r \in \mathbb{Z}_{>0}$, there exists an open covering $X = \cup U_i$ such that $\mathscr{F}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$ as $\mathcal{O}_X$-modules, moreover, we have

$$\{\text{locally free sheaves}\} \longleftrightarrow \{\text{vector bundles}\}$$

(2) The quotient of a locally free sheaf may by <u>NOT</u> locally free!

(2) Ideal sheaves

Definition 21.4 (Sheaf of ideals)**.** $(X, \mathcal{O}_X)$ a ringed space. A **sheaf of ideals** on X is a sheaf of $\mathcal{O}_X$-modules $\mathscr{I}$, which is a subsheaf of $\mathcal{O}_X$, i.e. $\forall U \subseteq X$ an open subset, $\Gamma(U, \mathscr{I})$, $\Gamma(U, \mathscr{I})$ is an ideal of $\Gamma(U, \mathcal{O}_X)$.

Example 21.5.

(1) $(X, \mathcal{O}_X)$ is a variety, $Z \subseteq X$ a closed subset. Define the sheaf of ideals $\mathscr{I}_Z$ associated to Z as following:

$$\Gamma(U, \mathscr{I}_Z) = \{s \in \Gamma(U, \mathcal{O}_X) | s|_{Z \cap U} \equiv 0\} \quad \text{for } \forall U \subseteq X \text{ an open subset}$$

(2) $X = \mathbb{A}_k^2$, $x = (0,0) \in X$. Consider the sheaf of ideal $\mathscr{I}_x$, (x_1, x_2) coordinate of $\mathbb{A}_k^2$, see that

$$\text{the stalk } (\mathscr{I}_x)_{x'} = \begin{cases} \mathcal{O}_{X,x'} & x' \neq x \\ \langle x_1, x_2 \rangle \cdot \mathcal{O}_{X,x} & x' = x \end{cases}$$

In particular, $\mathscr{I}_x$ is <u>NOT</u> locally free!

(3) $X = \mathbb{A}_k^2$, $x = (0,0) \in X$. Define a sheaf of ideal $\mathscr{I}$ as following:

- $x \notin U \subseteq X$ open subset, $\Gamma(U, \mathscr{I}) = \Gamma(U, \mathcal{O}_X)$.
- $x \in U \subseteq X$ open subset, $\Gamma(U, \mathscr{I}) = \{s \in \Gamma(U, \mathcal{O}_X) | s_x \in \mathfrak{m}_x^2\}$

$$\text{See that stalk } \mathscr{I}_{x'} = \begin{cases} \mathcal{O}_{X,x'} & x' \neq x \\ \langle x_1^2, x_1x_2, x_2^2\rangle \cdot \mathcal{O}_{X,x} & x' = x \end{cases}$$

(4) X an irreducible normal variety, D an effective Weil divisor. Recall we have defined $\mathcal{O}_X(-D)$ as following

$$\forall U \subseteq X \text{ open subset}, \Gamma(U, \mathcal{O}_X(-D)) = \{0 \neq \phi \in K(X) | \operatorname{div}(\phi)|_U + (-D)|_U \geq 0\} \cup \{0\}$$

As D is effective, hence $\Gamma(U, \mathcal{O}_X(-D)) \subseteq \Gamma(U, \mathcal{O}_X)$ is an ideal(you may use the property of DVR), and thus, $\mathcal{O}_X(-D)$ is a sheaf of ideals.

Fact 21.6. If $D = \sum_{i=1}^n D_i$ with D_i distinct prime divisor. Then $\mathcal{O}_X(-D) = \mathscr{I}_D$ as $\cup_{i=1}^n D_i$ is closed in X. Since for any open subset $U \subseteq X$, we have

$$\Gamma(U, \mathscr{I}_D) = \{s \in \Gamma(U, \mathcal{O}_X) | s|_{U\cap D} = 0\}$$

which means $\operatorname{div}(s)|_U - D|_U \geq 0$, hence $\mathscr{I}_D \subseteq \mathcal{O}_X(-D)$. On the other hand,

$$\Gamma(U, \mathcal{O}_X(-D)) = \{0 \neq \phi \in K(X) | \operatorname{div}(\phi)|_U - D|_U \geq\} \cup \{0\}$$

which means $\phi \in \Gamma(U, \mathcal{O}_X)$ and $\phi|_{D\cap U} = 0$, hence $\Gamma(U, \mathcal{O}_X(-D)) \subseteq \Gamma(U, \mathscr{I}_D)$.
In all, $\Gamma(U, \mathscr{I}_D) = \Gamma(U, \mathcal{O}_X(-D))$.

(3) Algebraic operations of sheaves of $\mathcal{O}_X$-modules

(a) **Tensor product**

$\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is the sheaf associated to the presheaf $U \mapsto \Gamma(U, \mathscr{F}) \otimes_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, \mathscr{G})$.

(b) **Direct sum**

The sheaf $\mathscr{F} \oplus \mathscr{G} : U \mapsto \Gamma(U, \mathscr{F}) \oplus \Gamma(U, \mathscr{G})$(it is already a sheaf).

(c) **Symmetric power**

For $m \in \mathbb{Z}_{>0}$. $\operatorname{Sym}^m \mathscr{F}$ is the shaef associated to the presheaf $U \mapsto \operatorname{Sym}^m_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, \mathscr{F})$.

(d) **Exterior power**

For $m \in \mathbb{Z}_{>0}$, $\wedge^m \mathscr{F}$ is the sheaf associated to the presheaf $U \mapsto \wedge^m_{\Gamma(U,\mathcal{O}_X)} \Gamma(U, \mathscr{F})$.

(e) **Hom-sheaf**

$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) : U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$.

(f) **Dual sheaf**

$\mathscr{F}^* := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$.

Remark 21.7.

(1) Taking stalks commutates with tensor product, direct sum, symmetric power, exterior power.

(2) $\mathscr{H}$om-sheaf is 'very bad' ! It does not commutate with taking stalk. In particular, we have $(\mathscr{F}^*)^* \neq \mathscr{F}$!

Recall 21.8. Recall that the constant sheaf $\underline{A}$ is the sheafification of the constant presheaf whose value is A, moreover, $\underline{A}_x$, the stalk of $\underline{A}$ at x, is A.

Example 21.9. $X = [0, 1]$ with cofinite topology. $\mathcal{O}_X = \underline{\mathbb{Z}}$ constant sheaf, $o \in X$.

(a) $\mathscr{F}$ = skyscraper sheaf $\mathbb{Z}_o$ at o, consider $\mathscr{H}om_{\underline{\mathbb{Z}}}(\mathbb{Z}_o|_U, \underline{\mathbb{Z}}|_U) = 0$left

(b) $\mathscr{F}, \mathscr{G}$ = extension of the constant sheaf $\underline{\mathbb{Z}}$ on $X \setminus \{o\}$. $\mathscr{F}_o = \mathscr{G}_o = 0$, hence $\mathrm{Hom}(\mathscr{F}_o, \mathscr{G}_o) = 0$. But $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \subseteq \mathrm{Hom}(\mathscr{F}, \mathscr{G})_o$.

(4) Base change

$f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ a morphism of ringed space.

(a) Let $\mathscr{F}$ be a $\mathcal{O}_X$-module. The **pull-back** $f^*\mathscr{F}$ is a sheaf of $\mathcal{O}_Y$-modules on Y defined as following:

  (i) $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ induces $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$ i.e. $\mathcal{O}_Y$ is a sheaf of $f^{-1}\mathcal{O}_X$-module.

  (ii) Then $\mathcal{O}_X$-module structure induces a $f^{-1}\mathcal{O}_X$-module structure of $f^{-1}\mathscr{F}$.

  Then $f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$.

  Caution 21.10. $f^* \neq f^{-1}$, $f^* = (\otimes_{f^{-1}\mathcal{O}_X}\mathcal{O}_Y) \circ f^{-1}$, where $\circ f^{-1}$ is an exact functor and $\otimes_{f^{-1}\mathcal{O}_X}\mathcal{O}_Y$ is NOT exact in general.

(b) **Push-forward**

  Let $\mathscr{G}$ be a $\mathcal{O}_Y$-module. For $\forall U \subseteq X$ open, $\Gamma(U, \mathcal{O}_X) \xrightarrow{f^\#} \Gamma(f^{-1}U, \mathcal{O}_Y)$ and $\Gamma(f^{-1}U, \mathscr{G})$ has a $\Gamma(f^{-1}U, \mathcal{O}_Y)$-module structure, see that $\Gamma(f^{-1}U, \mathscr{G}) = \Gamma(U, f_*\mathscr{G})$, hence it has a $\Gamma(U, \mathcal{O}_X)$-module structure, which means $f_*\mathscr{G}$ is a $\mathcal{O}_X$-module.

Proposition 21.11 (Exercise)**.** $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y) \xrightarrow{g} (Z, \mathcal{O}_Z)$ morphisms of ringed spaces.

(1) $(g \circ f)_* = g_* \circ f_*$.

(2) $(g \circ f)^* = f^* \circ g^*$.

Remark 21.12. $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ morphism of ringed spaces.

(1) **Pull-back of sections**

Let $\mathscr{F}$ be a $\mathcal{O}_X$-module. Then, $\forall U \subseteq X$ open, we have

$$\Gamma(U, \mathscr{F}) \to \Gamma(f^{-1}U, f^{-1}\mathscr{F}) \to \Gamma(f^{-1}U, f^*\mathscr{F})$$

by

$$s \mapsto s \mapsto s \otimes 1$$

In particular, we have $\Gamma(X, \mathscr{F}) \to \Gamma(Y, f^*\mathscr{F})$.

(2) Pull-back is compatible with direct sum, tensor product, exterior power, symmetric power, e.g.

$$f^*(\mathscr{F} \oplus \mathscr{G})\, f^*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G} = f^*\mathscr{F} \otimes_{\mathcal{O}_Y} f^*\mathscr{G})$$

But, pull-back is NOT compatible with $\mathscr{H}$om!!

(3) Push-forward is also 'very bad'. It is not compatible with algebraic operations.

21.2 §B. Quasi-coherent Sheaves on Affine Varieties

X = affine algebraic variety, $A = \Gamma(X, \mathcal{O}_X)$ the coordinate ring of X.

(1) Definition

Recall 21.13 (Localization of A-modules)**.** Let M be an A-module.

(1) $\forall x \in X$, define $M_x = M_{\mathfrak{p}_x} = M \otimes_A A_{\mathfrak{p}_x}$, where $\mathfrak{p}_x \subseteq A$ is the ideal of x i.e. $\mathfrak{p}_x = \{f \in A | f(x) = 0\}$.

(2) $0 \neq f \in A$, define $M_f = M \otimes_A A_f$.

Remark 21.14.

(1) The elements of M_x is a formal fraction

$$\left\{ \frac{m}{f} | m \in M, n \in \mathbb{Z}_{\geq 0}, f \in A \text{ and } f \notin \mathfrak{p}_x \right\} / \sim$$

where

$$\frac{m}{f} \sim \frac{m'}{f'} \iff \exists g \in A \text{ such that} g \in \mathfrak{p}_x \text{ and } g(f'm - fm') = 0 \text{ in } M.$$

(2) The elements of M_f is a formal fraction

$$\left\{ \frac{m}{f^n} | m \in M, n \in \mathbb{Z}_{\geq 0} \right\} / \sim$$

where

$$\frac{m}{f^n} \sim \frac{m'}{f^{n'}} \iff \exists r \in \mathbb{Z}_{\geq 0} \text{ such that} f^r(f^{n'}m - f^n m') = 0 \text{ in } M$$

Clearly, we have a natural morphism: $\forall 0 \neq f \in A$

$$M_f \to M_x, \quad x \in D(f) \subseteq X$$

by

$$\left[\frac{m}{f^n}\right] \mapsto \left[\frac{m}{f^n}\right]$$

Definition 21.15 ($\mathcal{O}_X$-module associated to M)**.** Let M be an A-module. Then the **$\mathcal{O}_X$-module $\widetilde{M}$ associated to** M is defined as following:

$$\forall U \subseteq X \text{ open, } U \mapsto \Gamma(U, \widetilde{M}) = \left\{ s : U \to \bigsqcup_{x \in U} M_x \right\}$$

where $s(x) \in M_x$ for any x. And for any $x \in U$, there exists $0 \neq f_x \in A$ such that $f_x(x) \neq 0$ and $m_x \in M$, $n_x \in \mathbb{Z}_{\geq 0}$ such that $s(x') = \frac{m_x}{f_x^{n_x}}$ for any $x' \in D(f) \cap U$.

Lemma 21.16.

(1) $\mathcal{O}_X = \widetilde{A}$ as A-module.

(2) For any $0 \neq f \in A$, $\Gamma\left(D(f), \widetilde{M}\right) = M_f$.

Definition 21.17. Let X be an affine algebraic variety. $A = \Gamma(X, \mathcal{O}_X)$, $\mathscr{F}$ a $\mathcal{O}_X$-module.

(1) $\mathscr{F}$ is called **quasi-coherent** if there exists an A-module M such that $\mathscr{F} \simeq \widetilde{M}$ as $\mathcal{O}_X$-modules.

(2) $\mathscr{F}$ is called **coherent** if there exists a finitely generated A-module M such that $\mathscr{F} \simeq \widetilde{M}$ as $\mathcal{O}_X$-modules.

(2) Algebraic operations of modules

Proposition 21.18. Let X be an affine algebraic variety. $A = \Gamma(X, \mathcal{O}_X)$.

(i) $\{M_i\}_{i \in I}$ a family of A-modules.
$(\widetilde{\oplus_{i \in I} M_i}) = \oplus_{i \in I} \widetilde{M_i}$

(ii) M, N two A-modules.
$(\widetilde{M \otimes_A N}) = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.

(iii) M an A-module, $p \in \mathbb{Z}_{>0}$.
$(\widetilde{\bigwedge^p M}) = \bigwedge^p \widetilde{M}$.

(iv) M an A-module, $p \in \mathbb{Z}_{>0}$.
$(\widetilde{\mathrm{Sym}^p M}) = \mathrm{Sym}^p\left(\widetilde{M}\right)$.

(v) $\phi : M \to N$ a homomorphism of A-modules $\rightsquigarrow \widetilde{\phi} : \widetilde{M} \to \widetilde{N}$ morphism of $\mathcal{O}_X$-modules. Then $\widetilde{\ker \phi} = \ker\left(\widetilde{\phi}\right)$, $\widetilde{\operatorname{im}(\phi)} = \operatorname{im}\left(\widetilde{\phi}\right)$, $\widetilde{\operatorname{coker}(\phi)} = \operatorname{coker}\left(\widetilde{\phi}\right)$.

Proposition 21.19. Let X be an affine algebraic variety. $A = \Gamma(X, \mathcal{O}_X)$. Let M, N be two A-modules. Then

$$\Gamma\left(X, \mathscr{H}om_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)\right) := \operatorname{Hom}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right) = \operatorname{Hom}_A(M, N)$$

In particular, if M is a finitely generated A-module, then $\widetilde{\operatorname{Hom}_A(M, N)} = \mathscr{H}om_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$, i.e. Hom commutes with sheafification of modules only if the source space is finitely generated!!

Proof.

- Given $\varphi \in \operatorname{Hom}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$, then $\varphi_X : \Gamma\left(X, \widetilde{M}\right) \to \Gamma\left(X, \widetilde{N}\right) \Rightarrow \varphi_X \in \operatorname{Hom}_A(M, N)$.
- Given $\varphi \in \operatorname{Hom}_A(M, N)$, then $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ is morphism of $\mathcal{O}_X$-modules, we have

  by

  $$\frac{m}{f^n} \mapsto \frac{\varphi(m)}{f^n}$$

  is a homomorpihsm of $\Gamma(D(f), \mathcal{O}_X)$-modules, which is A_f-modules.
- Assume that M is finitely generated. We need to show that $\widetilde{\operatorname{Hom}_A(M, N)} = \mathscr{H}om_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$. It is enough to show that $\mathscr{H}om_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$ is quasi-coherent. Then we need to prove that for any $0 \neq f \in A$, we have

□

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(3) Base change

Let $(f, f^{\#}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ be a morphism of affine algebraic varieties. $A = \Gamma(Y, \mathcal{O}_Y)$, $B = \Gamma(X, \mathcal{O}_X)$. We have

$$f^{\#} : B \to A$$

by

$$s \mapsto s \circ f$$

Proposition 22.1.

(1) Let M be an A-module. Then $f_* \widetilde{M} = \widetilde{M}$, where the first one is the sheafification as an A-module, the second one is the sheafification as an B-module.

(2) Let N be an B-module. Then $f^*\widetilde{N} = \widetilde{N \otimes_B A}$, where the first one is the sheafification as an B-module, the second is the sheafification as an A-module.

Remark 22.2. Let $f : Y \to X$ be a morphism of affine varieties.

(1) $\mathscr{F}$ is a auasi-coherent(resp. coherent) $\mathcal{O}_X$-module, then $f^*\mathscr{F}$ is quasi-coherent(resp. coherent).

(2) $\mathscr{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $f_*\mathscr{G}$ is quasi-coherent.

(3) But, even though $\mathscr{G}$ is coherent, $f_*\mathscr{G}$ may be NOT coherent, because if we have $B \to A$, M is a finitely generated A-module, we can not get M is a finitely generated B-module.

22.1 §C. Quasi-coherent and Coherent Sheaves on Varieties

Definition 22.3. Let $(X, \mathcal{O}_X)$ be a variety. A sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules is called **quasi-coherent**(resp. **coherent**) if

(1) there exists an affine open covering $X = \bigcup_{i \in I} U_i$.

(2) $\forall i \in I$, $\mathscr{F}|_{U_i}$ is quasi-coherent(resp. coherent).

Remark 22.4.

(1) $U \subseteq X$ open subset of a variety. $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $\mathscr{F}|_U := i^*\mathscr{F} = i^{-1}\mathscr{F}$, where $i : U \to X$ is the natural inclusion, see that both $i^*\mathscr{F}$ and $i^{-1}\mathscr{F}$ has a structure of $\mathcal{O}_X$-modules.

(2) Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules over a variety $(X, \mathcal{O}_X)$. Then $\mathscr{F}$ is quasi-coherent(resp. coherent) $\Longleftrightarrow$ for any $U \subseteq X$ affine open subset, $\mathscr{F}|_U$ is quasi-coherent(resp. coherent). In particular, if X is affine, then the definition above coincides with the one given in the previous section.

Idea of the proof: [[Har77] , II, Prop. 5.4] Without loss of generality, we may assume $X = U$ is affine. $X = \cup D(f_i)$ such that $\mathscr{F}|_{D(f_i)}$ is quasi-coherent(resp. coherent) for $0 \neq f_i \in A = \Gamma(X, \mathcal{O}_X)$. Set $M = \Gamma(X, \mathscr{F})$. Then there exists a natural morphism of $\mathcal{O}_X$-modules $\widetilde{M} \to \mathscr{F}$. It is enough to show

$$\Gamma\left(D(f_i), \widetilde{M}\right) = M_{f_i} \simeq \Gamma\left(D(f_i), \mathscr{F}\right)$$

Example 22.5.

(1) $\mathcal{O}_X$ is coherent.

(2) A locally free sheaf is coherent.

Proposition 22.6 (Closed under algebraic operators)**.** Let $(X, \mathcal{O}_X)$ be a variety.

(1) $\{F_i\}_{i\in I}$ a family of quasi-coherent sheaves, then $\oplus_{i\in I}\mathscr{F}_i$ is quasi-coherent.

(2) $\{\mathscr{F}_i\}_{i\in I}$ a family of coherent sheaves + I finite $\Rightarrow \oplus_{i\in I}\mathscr{F}_i$ is coherent.

(3) $\mathscr{F}, \mathscr{G}$ quasi-coherent(resp. coherent), then $\mathscr{F}\otimes\mathscr{G}$ is quasi-coherent(resp. coherent).

(4) $\mathscr{F}$ quasi-coherent(resp. coherent), then $\bigwedge^p\mathscr{F}$ and $\mathrm{Sym}^p\mathscr{F}$ is quasi-coherent(resp. coherent).

(5) $\phi : \mathscr{F} \to \mathscr{G}$ a morphism of quasi-coherent(resp. coherent) sheaves, then $\ker(\phi)$, $\mathrm{im}(\phi)$ and $\mathrm{coker}(\phi)$ are quasi-coherent(resp. coherent).

(6) $\mathscr{F}$ coherent, $\mathscr{G}$ quasi-coherent(resp. coherent), then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ is quasi-coherent(resp. coherent).

(7) $f : Y \to X$ a morphism. $\mathscr{F}$ is a quasi-coherent(resp. coherent) $\mathcal{O}_X$-module, then $f^*\mathscr{F}$ is quasi-coherent(resp. coherent).

(8) $f : Y \to X$ a morphism,. $\mathscr{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $f_*\mathscr{G}$ is quasi-coherent.

Proof. It follows from the properties over affine varieties. □

Example 22.7 (Push-forward of coherent sheaves)**.** $Y = \mathbb{A}^1_k$, $X = \mathrm{pt}$. $f : Y \to X$. Then $f_*\mathcal{O}_Y = \Gamma(Y, \mathcal{O}_Y) = k[T]$ and $\mathcal{O}_X = k$. But $k[T]$ is NOT a finitely generated k-module.

22.2 §D. Quasi-coherent and Coherent Sheaves on Projective Varieties

(1) From graded modules to quasi-coherent sheaves

Let $X \subseteq \mathbb{P}^n$ be a projective variety. Then the **homogeneous coordinate ring** of X is defined as

$$R := k[x_0, ..., x_N]/I(X)$$

where $I(X)$ is the homogeneous ideal of X, then R is a $\mathbb{Z}$-graded ring.

Remark 22.8. R depends on the embedding of X!! See for instance $\mathbb{P}^n$ and its Veronese embedding

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^N_k$$

Definition 22.9.

(1) A **graded R-module** M is a R-module with a decomposition $M = \oplus_{i\in\mathbb{Z}}M_i$ such that

$$S_d \cdot M_i \subseteq M_{i+d}, \quad d \in \mathbb{Z}_{\geq 0},\ i \in \mathbb{Z}$$

(2) Given a graded R-module M, we define a $\mathcal{O}_X$-module $\widetilde{M}$ as following: for $\forall 0 \neq F \in R$ a homogeneous element, then we have

$$X \overset{\text{open}}{\supseteq} D(F) \mapsto \Gamma\left(D(F), \widetilde{M}\right) = M_{(F)}$$

where

$$M_{(F)} = \left\{ \frac{m}{F^n} \middle| \deg m = n \cdot \deg(F) \right\} / \sim$$

and

$$\frac{m}{F^n} \sim \frac{m'}{F^{n'}} \Longleftrightarrow \exists r \in \mathbb{Z}_{\geq 0} \text{ such that } F^r\left(F^{n'}m - F^n m'\right) = 0 \text{ in } M$$

Proposition 22.10.

(1) $\mathcal{O}_X = \widetilde{R}$ as a graded R-module.

(2) For $\forall\ 0 \neq F \in R$ a homogeneous element, then

$$\widetilde{M}|_{D(F)} = \widetilde{M_F}$$

where view $\widetilde{M_{(F)}}$ as a $R_{(F)}$-module, and $R_{(F)} = \Gamma\left(D(F), \mathcal{O}_X\right)$.

(3) M is a finitely generated R-module, then $\widetilde{M}$ is coherent.

Remark 22.11. The converse of (3) is NOT true in general.

Example 22.12. Let $Y \overset{i}{\subseteq} X \subseteq \mathbb{P}_k^N$ be two projective varieties with R_X and R_Y the homogeneous coordinate rings, respectively. Then there exists a natural suejective homomorphism $R_X \to R_Y$. Let I_Y be the kernel. Then we have

$$0 \longrightarrow I_Y \longrightarrow R_X \longrightarrow R_Y \longrightarrow 0$$

as R_X-modules. Then it induces an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{I_Y} \longrightarrow \widetilde{R_X} \longrightarrow \widetilde{R_Y} \longrightarrow 0$$

which is exactly

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0$$

Lemma 22.13. Let $\varphi : M \to N$ be a homomorphism preserving degrees bwtween R-modules. Assume $\varphi_n : M_n \to N_n$ is surjective for $\forall\ n >> 0$, then $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ is a surjective morphism of sheaves.

Proof. For $0 \neq F \in R$ a homogeneous element, it is enough to show

$$\varphi_{(F)} : M_{(F)} \to N_{(F)}$$

is surjective. Consider $\frac{y}{F^r} \in N_{(F)}$, then there exists $s \in \mathbb{Z}_{>0}$ such that $F^s y \in \operatorname{im}(\varphi)$ and hence

$$\frac{y}{F^r} = \frac{F^s \cdot y}{F^{s+r}} \in \operatorname{Im}\left(\varphi_{(F)}\right)$$

□

(2) From quasi-coherent sheaves to graded modules

Definition 22.14. Let M be a graded R-module. Given $d \in \mathbb{Z}$, then the graded R-module M, $M(d)$ is defined to be M with the shifted grading $M(d)_n := M_{n+d}$.

Definition 22.15. Let $X \subseteq \mathbb{P}^N_k$ be a projective variety with homogeneous coordinate ring R and $i : X \to \mathbb{P}^N_k$.

(1) $\mathcal{O}_X(d) := \mathcal{O}_{\mathbb{P}^N_k}(d)|_X := i^*\mathcal{O}_{\mathbb{P}^N_k}(d)$ is the sheaf associated to $R(d)$.

(2) If $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $\mathscr{F}(d) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$.

Remark 22.16.

(1) $\mathcal{O}_X(d) = i^*\mathcal{O}_{\mathbb{P}^N_k}(d)$ over $D(F)$, $\Gamma\left(D(F), \mathcal{O}_X(d)\right) =$ homogeneous of degree d in R_F, i.e. $\frac{G}{F^n}$ with $\deg G = d + n \cdot \deg F$.

(2) If M is a graded R-module, then $\widetilde{M(d)} = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)$.

Definition 22.17. Let $\mathscr{F}$ be a quasi-oherent sheaf on a projective variety $X \subseteq \mathbb{P}^N_k$. Define a graded R-module as $\Gamma_*\mathscr{F} := \oplus_{d \in \mathbb{Z}} \Gamma\left(X, \mathscr{F}(d)\right)$ with module structure coming from

$$\Gamma\left(X, \mathcal{O}_X(d)\right) \otimes \Gamma\left(X, \mathscr{F}(d')\right) \to \Gamma\left(X, \mathscr{F}(d + d')\right)$$

Remark 22.18. $\Gamma_*\mathscr{F}$ depends on the embedding of X into $\mathbb{P}^N_k$!!

Proposition 22.19. Let $\mathscr{F}$ be a quasi-coherent sheaf on a projective variety $X \subseteq \mathbb{P}^N_k$.

(1) $\widetilde{\Gamma_*\mathscr{F}} \simeq \mathscr{F}$ as $\mathcal{O}_X$-modules.

(2) $\mathscr{F}$ is coherent if and only if $\Gamma_*\mathscr{F}$ is a finitely generated R-module.

Remark 22.20.

(1) If $\mathscr{F} = \widetilde{M}$ for some graded R-module M, then M may be <u>NOT</u> isomorphic to $\Gamma_*\mathscr{F}$.

(2) there exists a natural homomorphism

$$\Gamma_d : M_d \to \Gamma\left(X, \widetilde{M(d)}\right)$$

by

$$m \mapsto \frac{m}{1}$$

for all $d \in \mathbb{Z}$. But, ingeneral, Γ_d is neither injective nor surjective.

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23.1 §E. Back to Locally Free Sheaves

Let $(X, \mathcal{O}_X)$ be affine algebraic variety. A $\mathcal{O}_X$-module $\mathscr{F}$ is **locally free** of rank r if there exists an open covering $X = \cup_{i \in I} U_i$ such that $\mathscr{F}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r}$.

Remark 23.1.

(1) We may assume that U_i's are affine.

(2) Locally free sheaves are coherent as $\mathscr{F}|_{U_i} = \widetilde{A_i^{\oplus r}}$, $A_i = \Gamma(U_i, \mathcal{O}_{U_i})$, U_i affine.

(1) Relation with projective modules

Recall 23.2 (Projective A-modules)**.** An A-module M is **projective** if $\forall$ surjective homomorphism of A-modules $g : P \to N$ and a homomorphism of A-modules $h : M \to N$, there exists $\overline{h} : M \to \P$ such that $h = g \circ \overline{h}$.

$$\begin{array}{ccccc} & & M & & \\ & \exists\overline{h} \swarrow & \downarrow & & \\ P & \xrightarrow[g]{} & N & \longrightarrow & 0 \end{array}$$

Proposition 23.3. Let $(X, \mathcal{O}_X)$ be an affine algebraic variety, let $A = \Gamma(X, \mathcal{O}_X)$ and let M be a finitely generated A-module. Then $\widetilde{M}$ is locally free or rank r if and only if one of following holds:

(1) there exists $f_1, ..., f_m \in A$ such that M_{f_i} is a free A_{f_i}-module of rank r and $(1) = \langle f_1, ..., f_m \rangle$.

(2) $\forall x \in X$, $\widetilde{M}_x$ is a free $\mathcal{O}_{X,x}$-module of rank r.

(3) M is a projective A-module.

Proof. See [D · Eisenbud, GTM 150, Thm A.3.2]. □

(2) Fibres and Nakayama's lemma

Definition 23.4. Let $(X, \mathcal{O}_X)$ be an algebraic variety, $\mathscr{F}$ is a coherent sheaf over X, $x \in X$ a point. The **fibre** of $\mathscr{F}$ at x is defined as

$$\mathscr{F}(x) := \mathscr{F}_x / \mathfrak{m}_x \mathscr{F}_x$$

where $\mathscr{F}_x$ is the stalk of $\mathscr{F}$ at x and $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$.

Remark 23.5.

(1) $i_x : x \hookrightarrow X$ the natural inclusion. Then $\mathscr{F}(x) = i_x^* \mathscr{F}$? the left is an algebra, the right is a sheaf

(2) $\mathscr{F}(x)$ is a k-vector space given as following:

$$\mathcal{O}_{X,x}/\mathfrak{m}_x \simeq k \to \mathscr{F}(x) = \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}(x)$$

$\dim_k \mathscr{F}(x) < +\infty$ as $\mathscr{F}$ is coherent. e.g. let $\mathscr{F}$ be the sheaf of sections of a vector bundle V of rank r. Then

(a) $\mathscr{F}(x) = V_x =$ fibre of V over $x = k^r$.

(b) $\mathscr{F}_x = \mathcal{O}_{X,x}^{\oplus r}$

Proposition 23.6 (Nakayama's lemma)**.** Let $(X, \mathcal{O}_X)$ be an algebraic variety, and let $\mathscr{F}$ be a coherent sheaf over X. Then the followings are equivalent:

(1) $\mathscr{F}$ is locally free of rank r.

(2) $\mathscr{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank r, for $\forall x \in X$.

(3) $\mathscr{F}(x)$ is a k-vector space of dimension r, for $\forall x \in X$.

Recall 23.7 (Nakayala lemma)**.** Let A be a local ring, $\mathfrak{m} =$ maxiaml ideal of A. M is a finitely generated A-module. If $m_1, ..., m_r \in M$ are elements in M such that $\overline{m_1}, ..., \overline{m_r} \in M/\mathfrak{m}M$ form a $A/\mathfrak{m}$-basis, then $m_1, ..., m_r$ form a A-basis for M, e.g. any $A/\mathfrak{m}$-basis of $M/\mathfrak{m}M$ can be lifted to be a basis of M over A.

(3) Morphisms of vector bundles vs morphisms of locally free sheaves

Let V_1, V_2 be two vectore bundles over an algebraic variety X. Let $\mathscr{F}_1$, $\mathscr{F}_2$ be sheaves of sections of V_1 and V_2, respectively. Then there exists a natural injection

$$\mathrm{Hom}_{\mathrm{vect}}(V_1, V_2) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_2)$$

where the left is the homomorphism of vector bundles and the right is the homomorphism of $\mathcal{O}_X$-modules.

Lemma 23.8. Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}_1, \mathscr{F}_2)$. Then $\varphi \in \mathrm{Hom}_{\mathrm{vect}}(V_1, V_2) \iff \forall\ x \in X$, $\varphi(x) : \mathscr{F}_1(x) \to \mathscr{F}_2(x)$ is of canstant rank, i.e. the rank of the k-liner map is independent of $x \in X$.

Example 23.9. $V_1 = X \times k$, $V_2 = X \times k$, $X = \mathbb{A}_k^1$. Take

$$\varphi : \mathcal{O}_X \to \mathcal{O}_X$$

by

$$s \mapsto x \cdot s$$

At $x = 0$, $\operatorname{rank} \varphi(0) = 0$, and $\operatorname{rank} \varphi(x) = 1$ for $x \neq 0$.

(4) Pull-back of locally free sheaves

Recall 23.10. Recall that for $f : Y \to X$, $f^* = \otimes_{f^{-1}\mathcal{O}_X}\mathcal{O}_Y \circ f^{-1}$, where $\otimes_{f^{-1}\mathcal{O}_X}\mathcal{O}_Y$ is right exact and f^{-1} is exact.

Proposition 23.11. Let $f : Y \to X$ be a morphism of varieties.

(1) (Compatible with pull-back of vector bundles).

Let V be a vector bundle over X. Let $\mathscr{F}$, $\mathscr{F}'$ be the sheaves of sections of V and f^*V, respectively. Then $\mathscr{F}' \simeq f^*\mathscr{F}$.

(2) (Compatible with pull-back of Cartier divisors)

Assume f is dominant and both Y and X are irreducible normal varities. Let D be a Cartier divisor on X. Then $f^*\mathcal{O}_X(D) \simeq \mathcal{O}_Y(f^*D)$. It is NOT true for D being a Weil divisor because $\mathcal{O}_X(D)$ is NOT locally free in general.

(3) Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be a short exact sequence of quasi-coherent sheaves on X.

(3.a) Let $\mathscr{F}$ be another quasi-coherent sheaf. Then the following sequence

$$0 \longrightarrow \mathscr{F}_1 \otimes \mathscr{F} \longrightarrow \mathscr{F}_2 \otimes \mathscr{F} \longrightarrow \mathscr{F}_3 \otimes \mathscr{F} \longrightarrow 0$$

is exact if either $\mathscr{F}$ is locally free of all $\mathscr{F}_i$ are locally free.

(3.b) If all $\mathscr{F}_i$'s are locally free, then

$$0 \longrightarrow f^*\mathscr{F}_1 \longrightarrow f^*\mathscr{F}_2 \longrightarrow f^*\mathscr{F}_3 \longrightarrow 0$$

is exact.

Proof. (1) and (2) follows from the definition.
(3) follows from the following fact:
Let A be a Noetherian ring. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of A-modules. Let M be another A-module. Then the following sequence

$$0 \longrightarrow M_1 \otimes M \longrightarrow M_2 \otimes M \longrightarrow M_3 \otimes M \longrightarrow 0$$

is exact if either M is a free A-module of finite rank or all M_i are free A-modules of finite rank. $\square$

(5) Relations between coherent sheaves and locally free sheaves

Proposition 23.12. Let $\mathscr{F}$ be a coherent sheaf over a variety X.

(1) (Generic freeness)

There exists $\varnothing \neq U \subseteq X$ open subset such that $\mathscr{F}|_U$ is locally free.

(2) (Serre)

If $X \subseteq \mathbb{P}_k^N$ is a projective variety, then there exists finitely many $n_i \in \mathbb{Z}$ with a surjective morphism

$$\oplus_i \mathcal{O}_X(n_i) \twoheadrightarrow \mathscr{F}$$

i.e. $\mathscr{F}$ is a quotient of some locally free sheaf over X.

Idea of the proof of (2): Find $n >> 0$ such that $\mathscr{F}(n)$ is globally generated i.e. $\Gamma(X, \mathscr{F}(n)) \otimes \mathcal{O}_X \to \mathscr{F}(n)$ is surjective. Then $r = \dim_k \Gamma(X, \mathscr{F}(n)) < +\infty$, hence $\oplus_{i=1}^r \mathcal{O}_X(-n) \twoheadrightarrow \mathscr{F}$.

23.2 §F. Differectials and Cotangent Sheaf

(1) Käler differential

Definition 23.13. Let R be a k-algebra. We define Ω_R to be the free R-module generated by the formal symbols df for all $f \in R$, modulo the relations:

(1) $d(f+g) = df + dg$, $\forall\ f, g \in R$.

(2) $d(f \cdot g) = fdg + gdf$, $\forall\ f, g \in R$.

(3) $df = 0$, $\forall\ f \in k$.

Then elements of Ω_R are called **(Kähler) differentials** of R(over k).

Example 23.14.

(1) $R = k[x_1, ..., x_n]$. Then $df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$ by (1) + (2) + (3) for $\forall\ f \in R$. Hence $\Omega_R = Rdx_1 \oplus \cdots \oplus Rdx_n$ and we can regard Ω as liner forms on the Zariski tangent space $T_x\mathbb{A}_k^n$ of $\mathbb{A}_k^n$ which depend on x algebraically. More precisely,

$$T_{\mathbb{A}_k^n}^{Zar} = \mathbb{A}_k^n \times k^n \xrightarrow{df} k$$

by

$$(x, (v_1, ..., v_n)) \mapsto \frac{\partial f}{\partial x_1}(x) \cdot v_1 + \cdots \frac{\partial f}{\partial x_n}(x) \cdot v_h$$

(2) In general, let $R = k[x_1, ..., x_n]/I(V)$ be the coordinate ring of affine algebraic variety $V \subseteq \mathbb{A}_k^n$. Assume that $I(V) = \langle F_1, ..., F_r \rangle$. Then

$$\Omega_R = (Rdx_1 \oplus \cdots \oplus Rdx_n) / \langle dF_1, ..., dF_r \rangle$$

and the elements of Ω_R can be viewed as linear maps defined on T_pV, $p \in V$ which depend on p algebracially,

$$T_V^{Zar} \subseteq V \times k^n \xrightarrow{df} k \quad f \in R$$

by

$$(p,(v_1,...,v_n)) \mapsto \frac{\partial f}{\partial x_1}(p)\cdot v_1 + \cdots \frac{\partial f}{\partial x_n}(p)\cdot v_h$$

Recall from [Chap IV. §C.], for $\forall\ f \in I(V) = \langle F_1, ..., F_r\rangle$, then

$$df\Big|_{T_V^{Zar}} \equiv 0$$

hence $\Omega_R = Rdx_1 \oplus \cdots \oplus Rdx_n / \langle df,\ f \in I(V)\rangle = Rdx_1 \oplus \cdots \oplus Rdx_n/\langle dF_i,\ 1 \le i \le r\rangle$.

Let $p \in V$ be a point with $\mathfrak{m}_p \subseteq I(V) \subseteq R$ be the maximal ideal. Then

$$\Omega_R \otimes_R R/\mathfrak{m} \simeq (kdx_1 \oplus \cdots \oplus kdx_n) \Big/ \left\langle \sum_{j=1}^{n} \frac{\partial F_i}{\partial x_j}(p)dx_j,\ 1 \le i \le r \right\rangle$$

is a k-vector space with dimension $\dim(T_pV - p)$ and we can regard $\Omega_R \otimes_R R/\mathfrak{m}$ as $\mathrm{Hom}_k\left(T_pV - p, k\right) = \Omega_{V,p}$.

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22/11/28.

References for differentials:

1. [Har77] , II, §8.
2. [Mus] , §8.7.

Reference for generic freeness:

H. Matsumura. Commutative algebra, lemma 22.1.

(2) Cotangent sheaf of $\mathbb{P}^n$

Proposition 24.1 (Euler sequence)**.** For $n \in \mathbb{Z}_{>0}$, the cotangent sheaf $\Omega_{\mathbb{P}^n}$ of $\mathbb{P}^n$ is determined by an exact sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \xrightarrow{\ f\ } \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \xrightarrow{\ g\ } \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Proof. Step 1. Construction of f.

Consider $i, j \in \{0, 1, ..., n\}$ with $i \neq j$. Then the regular function $\frac{x_i}{x_j} \in \Gamma\left(U_j, \mathcal{O}_{\mathbb{P}^n}\right)$, where $U_j = D(x_j)$. Then Left

□

Remark 24.2. Taking dual yields

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \longrightarrow \Omega_X^* \longrightarrow 0$$

then tensoring with $\mathcal{O}_{\mathbb{P}^n}(-1)$ yields:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\widetilde{f}} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow \Omega_X^* \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow 0$$

where $\widetilde{f}$ is the natural morphism in the definition of $\mathcal{O}_{\mathbb{P}^n}(-1)$, i.e.

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \left\{([l], v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid v \in l\right\} \subseteq \mathbb{P}^n \times \mathbb{C}^{n+1}.$$

and $\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}$ is the trivial vector bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$.

(3) Canonical bundle

Definition 24.3. Let X be a variety, $n = \dim X$.

(1) The tangent sheaf $\mathcal{T}_X$ is defined as Ω_X^*.

(2) If X is irreducible and normal, we define the canonical bundle ω_X as $\left(\bigwedge^n \Omega_X\right)^{**}$

Remark 24.4.

(1) In general, $\mathcal{T}_X^* \neq \Omega_X$. We lose many informations of X by taking dual of Ω_X. However, if X is nonsigular and irreducible, then $\mathcal{T}_X^* = \Omega_X = \Omega_X^{**}$ and $\mathcal{T}_X$ is nonsigular which is the sheaf of sections of the usual tangent bundle.

(2) In general, ω_X is not locally free, by=ut as X is normal and irreducible, thus there exists $Z \subseteq X$ of $\operatorname{codim} Z \geq 2$ such that $\omega_X|_U = \bigwedge^n \Omega_X|_U = \bigwedge^n \Omega_U = \det(\Omega_X)$.

Example 24.5 (Canonical bundle of $\mathbb{P}^n$).

Fact 24.6. Let $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$ be a short exact sequence of locally free sheaves. Then $\det(\mathscr{E}) \simeq \det(\mathscr{F}) \otimes \det(\mathscr{G})$.

Appling it to Euler sequence of $\mathbb{P}^n$:

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

We have $\det\left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}\right) \simeq \omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}$, and thus $\mathcal{O}_{\mathbb{P}^n}(-n-1) \simeq \omega_{\mathbb{P}^n}$.

IX Cohomology of Coherent Sheaves

24.1 §A. Cěch Cohomology

(1) Motivation for shef cohomology

(a) For a short exact sequence $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ of sheaves on a topological space X. The induced morphisms

$$0 \longrightarrow \Gamma(X, \mathscr{F}_1) \longrightarrow \Gamma(X, \mathscr{F}_2) \longrightarrow \Gamma(X, \mathscr{F}_3)$$

We denote it by (*) in short, (*) may be NOT surjective at the right hand side in general. So we can NOT get much informations about $\Gamma(X, \mathscr{F}_3)$ by (*). Cohomology gives a natural way to extend this sequence to the right. We will construct naturally defined groups $H^i(X, \mathscr{F})$ for any sheaf $\mathscr{F}$ on X and $i \in \mathbb{Z}_{\geq 0}$ such that there is a long exat sequence

$$\begin{aligned} 0 &\to \Gamma(X, \mathscr{F}_1) \to \Gamma(X, \mathscr{F}_2) \to \Gamma(X, \mathscr{F}_3) \\ &\to H^1(X, \mathscr{F}_1) \to H^1(X, \mathscr{F}_2) \to H^1(X, \mathscr{F}_3) \\ &\to H^2(X, \mathscr{F}_1) \to H^2(X, \mathscr{F}_2) \to H^2(X, \mathscr{F}_3) \\ &\to \cdots \end{aligned}$$

(b) If X is a variety and $\mathscr{F}$ is a coherent sheaf, then $H^i(X, \mathscr{F})$ is a k-vector spavce. Hence, apply this to the canonically defined coherent sheaves on X, e.g. $\mathcal{O}_X, \Omega_X, \mathcal{T}_X, \omega_X ...$ The dimensions of the cohomological groups of them are important intrinsic invariants of X which can be used to distinguish varieties.

(2) Cěch Cohomology

Let X be a topological space. $\mathscr{U} = (U_i)_{i \in I}$ is an open covering of X. I is a well-ordered index set. Let $\mathscr{F}$ be a sheaf of abelian groups on X. $(i_0, ..., i_p) \in I^{p+1}$, the intersection of U_{i_j} for $i = 0, 1, ..., p$, is denoted by $U_{i_0 \dots i_p}$.

Definition 24.7. For each $p \geq 0$, we define

$$C^p(\mathscr{U}, \mathscr{F}) = \prod_{i_0 < \dots < i_p} \Gamma(U_{i_0 \dots i_p}, \mathscr{F})$$

i.e. if $\alpha \in C^p(\mathscr{U}, \mathscr{F})$, then we have $\alpha = (\alpha_{i_0 \dots i_p})_{i_0 < \dots < i_p}$ with $\alpha_{i_0 \dots i_p} \in \Gamma(U_{i_0 \dots i_p}, \mathscr{F})$.

We define the coboundary map

$$\partial_p : C^p(\mathscr{U}, \mathscr{F}) \to C^{p+1}(\mathscr{U}, \mathscr{F})$$

by

$$\alpha = (a_{i_0 \dots i_p})_{i_0 < \dots < i_p} \mapsto \left(\sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \widehat{i_k} \dots i_{p+1}} |_{U_{i_0 \dots i_{p+1}}} \right)_{i_0 < \dots , i_p}$$

Example 24.8. $X = U_0 \cup U_1 \cup U_2$.
$C^0(\mathscr{U}, \mathscr{F}) = \{S = (S_0, S_1, S_2) | S_i \in \Gamma(U_i, \mathscr{F})\}$.
$C^1(\mathscr{U}, \mathscr{F}) = \{(S_{01}, S_{02}, S_{12}) | S_{ij} \in \Gamma(U_i \cap U_j, \mathscr{F})\}$.
$C^2(\mathscr{U}, \mathscr{F}) = \{S_{012} | S_{012} \in \Gamma(U_0 \cap U_1 \cap U_2, \mathscr{F})\}$.
$\partial_0(S) = (S_{01}, S_{02}, S_{12})$, where
$S_{01} = (-1)^0 S_1|_{U_0 \cap U_1} + (-1)^1 S_0|_{U_0 \cap U_1} = (S_1 - S_0)|_{U_0 \cap U_1}$.
$S_{02} = (-1)^0 S_2|_{U_0 \cap U_2} + (-1)^1 S_0|_{U_0 \cap U_2} = (S_2 - S_0)|_{U_0 \cap U_2}$.
$S_{12} = (-1)^0 S_2|_{U_1 \cap U_2} + (-1)^1 S_1|_{U_1 \cap U_2} = (S_2 - S_1)|_{U_1 \cap U_2}$.

Lemma 24.9. $\partial^2 = 0$.

$C^\bullet(\mathscr{U}, \mathscr{F})$ = complex obatined as above.

Definition 24.10. Let X be a topological space and let $\mathscr{U}$ be an open covering of X. For any sheaf of abelian groups $\mathscr{F}$ on X, we define the p-th Cěch cohomology group of $\mathscr{F}$ with respect to the cover to be

$$\check{H}^p(\mathscr{U}, \mathscr{F}) := H^p(C^\bullet(\mathscr{U}, \mathscr{F})) = \frac{\operatorname{Ker} \partial_p}{\operatorname{Im} \partial_{p-1}}$$

Proposition 24.11. The canonical map $\Gamma(X, \mathscr{F}) \to \check{H}^0(\mathscr{U}, \mathscr{F})$ is an isomorphism.

Proof. Just by definition of sheaf. □

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22/11/30.
References for cohomologies:

(1) [Har77] , III, §1-5, §7.

(2) Mustată, Chapter 10, Chapter 14.

25.1 §B. Brief Introduction of Right Derived Functors

Definition 25.1. Let $\mathscr{A}$ be an abelian category.

(1) An object $I \in \mathscr{A}$ is **injective** if the functor $\operatorname{Hom}(-, I)$ is exact.

(2) Let $A \in \mathscr{A}$ be an object. An **injective resolution** of A is an exact complex

$$0 \longrightarrow A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^m \longrightarrow \cdots$$

such that $I^m (m \geq 0)$ are injective objects.

(3) Let $F : \mathscr{A} \to \mathscr{B}$ be a covariant left exact functor of categories with $\mathscr{A}$ is an abelian category **having enough injective objects**(every object in $\mathscr{A}$ has an injective resolution). Then the **right derived functor** $R^iF : \mathscr{A} \to \mathscr{B}$ is a functor such that if $0 \to A \xrightarrow{\varepsilon} I^\bullet$ is an injective resolution of an object $A \in \mathscr{A}$, then $R^iF(A) = H^i(F(I^\bullet))$ for $i \geq 0$.

$$H^i(F(I^\bullet)) = \frac{\operatorname{Ker}(F(d^i))}{\operatorname{Im}(F(d^{i-1}))}$$

and we have

$$0 \longrightarrow F(A) \xrightarrow{F(\varepsilon)} F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} \cdots \longrightarrow F(I^m) \longrightarrow \cdots$$

where is exact at $F(I^0)$.

Remark 25.2.

(1) $R^i F(A)$ does not depend on the resolution $0 \to A \xrightarrow{\varepsilon} I^\bullet$.

(2) Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category of sheaves of $\mathcal{O}_X$-modules has enough injectives.

Definition 25.3.

(1) Let (X, MOX) be a variety. Then $H^i(X, -)$ is the right derived functor of $\Gamma(X, -)$, i.e. $\forall \mathscr{F}$ a quasi-coherent sheaf on X, taking an injective resolution $0 \to \mathscr{F} \xrightarrow{\varepsilon} \mathscr{I}_0 \xrightarrow{d^0} \mathscr{I}_1 \xrightarrow{d^1} \mathscr{I}_2 \to \cdots$. Then

$$H^i(X, \mathscr{F}) = \frac{\operatorname{Ker}(\Gamma(d^i))}{\operatorname{Im}(\Gamma(d^{i-1}))}$$

and we have

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \xrightarrow{\Gamma(\varepsilon)} \Gamma(X, \mathscr{I}_0) \xrightarrow{\Gamma(d^0)} \Gamma(X, \mathscr{I}_1) \xrightarrow{\Gamma(d^1)} \Gamma(X, \mathscr{I}_2) \xrightarrow{\Gamma(d^2)} \cdots$$

which is exact at $\Gamma(X, \mathscr{I}_0)$.

(2) Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism between varieties. Then $R^i f_*$ is the right derived functor of f_*, which are called the **higher direct images**.

Remark 25.4.

(1) $H^i = R^i\Gamma(X, -)$ is a special case of $R^i f_*(-)$: taking $f : X \to \mathrm{pt}$ to be a constant map to a point. Then $f_*(-) = \Gamma(X, -)$.

(2) $H^0(X, \mathscr{F}) = \Gamma(X, \mathscr{F})$.

25.2 §C. Comparing Cěch Cohomologies and Sheaf Cohomologies

(1) Serre's Theorem A and Theorem B

Theorem 25.5 (Serre, [Har77] , III, §3)**.** *Let X be an affine algebraic variety, $\mathscr{F}$ a quasi-coherent sheaf on X. Then*

(1) $\mathscr{F}$ is globally generated.

(2) $H^q(X, \mathscr{F}) = 0$ for $\forall\ q > 0$.

proof is left

(2) Leray's Theorem

Definition 25.6. Let $\mathscr{F}$ be a sheaf of abelian groups over a topological space X. Let $\mathscr{U} = (U_i)_{i\in I}$ be an open covering of X. Then $\mathscr{U}$ is called **$\mathscr{F}$-acyclic** if for $\forall\ q > 0$, $\forall\ p \geq 0$, and $\forall\ (i_0, ..., i_p) \in I^{p+1}$, we have

$$H^q(U_{i_0\cdots i_p}, \mathscr{F}|_{U_{i_0\cdots i_p}}) = 0$$

Theorem 25.7 (Leray)**.** *Let $\mathscr{F}$ be a sheaf of abelian groups over a topological space X. Let $\mathscr{U} = (U_i)_{i\in I}$ be an open covering of X.*

(1) There exists a natural canonical functorial morphism, $\forall q \geq 0$.

$$\check{H}^q(\mathscr{U}, \mathscr{F}) \to H^q(X, \mathscr{F})$$

(2) If the covering $\mathscr{U}$ is $\mathscr{F}$-acyclic, then this morphism $(\star)$ is an isomorphism.

Proof. [Har77] , III, Lemma 4.4 and Theorem 4.5 □

Corollary 25.8. Let X be affine algebraic variety, $\mathscr{U}$ an affine open covering of X, $\mathscr{F}$ a quasi-coherent sheaf of X. Then we have a natural canonical functorial isomorphism

$$\check{H}^q(\mathscr{U}, \mathscr{F}) \to H^q(X, \mathscr{F})$$

for all $q \in \mathbb{Z}_{\geq 0}$.

Proof. $U, V \subseteq X$ be affine open subsets $\Rightarrow U \cap V$ is an affine open subset of X(by separateness), so $\mathscr{U}$ is $\mathscr{F}$-acyclic. □

25.3 §D. Calculation by Cěch Cohomology: an example

In this section, we aim to use Leray's Theorem to calculate the cohomologies of $\mathcal{O}_X$, where $X = \mathbb{A}_k^n \setminus \{0\}$.

(1) Künneth formula

Let $K^\bullet$ and $L^\bullet$ be two complexes of vector spaces over k. We define the complex $K^\bullet \otimes L^\bullet$ by setting

(1) $(K^\bullet \otimes L^\bullet)^n = \bigoplus_{p+q=n} K^p \otimes L^q$.

(2) $d(x \otimes y) = dx \otimes y + (-1)^p x \otimes dy$, where $x \in K^p$, $y \in L^q$.

The complex $K^\bullet \otimes L^\bullet$ is called the **tensor product of $K^\bullet$ and $L^\bullet$**. We have a canonical linear map

$$\mu : \bigoplus_{p+q=n} H^p(K^\bullet) \otimes H^q(L^\bullet) \to H^n(K^\bullet \otimes L^\bullet)$$

defined as

$$\mu : ([x] \otimes [y]) \mapsto [x \otimes y]$$

Proposition 25.9. The linear map μ is an isomorphism.

proof is left

Remark 25.10 (Künneth theorem, general form)**.** Let X, Y be two varieties and let $\mathscr{F}, \mathscr{G}$ be quasi-coherent sheaves on X and Y respectively. Then

$$H^n(X \times Y, p_X^*\mathscr{F} \otimes p_Y^*\mathscr{G}) = \bigoplus_{p+q=n} H^p(X, \mathscr{F}) \otimes H^q(Y, \mathscr{G})$$

where

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_Y} & Y \\ \downarrow{\scriptstyle p_X} & & \\ X & & \end{array}$$

Hint: Eilenberg-Zilber theorem.

(2) Calculation of $H^q(\mathbb{A}^2_k \setminus \{0\}, \mathcal{O})$
Let $X = \mathbb{A}^2_k \setminus \{0\}$ and let (x_0, x_1) be the coordinate of $\mathbb{A}^2_k$. $U_0 = D(x_0) \subseteq X$. Then $X = U_0 \cup U_1$ and U_0, U_1 are both aavs.

$$0 \longrightarrow C^0(\mathscr{U}, \mathcal{O}_X) = \Gamma(U_0, X) \oplus \Gamma(U_1, \mathcal{O}_X) \longrightarrow \Gamma(U_0 \cap U_1, \mathcal{O}_X) \longrightarrow 0$$

by

$$(f, g) \mapsto (f - g)|_{U_0 \cap U_1}$$

Consider the complex $K^\bullet$ defined as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (K^\bullet \otimes K^\bullet)^0 & \longrightarrow & (K^\bullet \otimes K^\bullet)^1 & \longrightarrow & (K^\bullet \otimes K^\bullet)^2 & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \Gamma(X, \mathcal{O}_X) & \longrightarrow & C^0(\mathscr{U}, \mathcal{O}_X) & \longrightarrow & C^1(\mathscr{U}, \mathcal{O}_X) & \longrightarrow & 0 \\ & & \text{Hartogs}\| & & \| & & \| & & \\ 0 & \longrightarrow & \Gamma(\mathbb{A}^2_k, \mathcal{O}_{\mathbb{A}^2_k}) & \longrightarrow & \Gamma(U_i, \mathcal{O}_X) & \longrightarrow & \Gamma(U_0 \cap U_1, \mathcal{O}_X) & \longrightarrow & 0 \end{array}$$

See that $\Gamma(\mathbb{A})$

Left

(3) General case: $X = \mathbb{A}_k^{n+1} \setminus \{0\}$

Consider the affine open covering $\mathscr{U} = (U_i)_{0 \geq i \geq n}$, $U_i = D(x_i) \subseteq X$. Then we have

$$H^q(X, \mathcal{O}_X) = \check{H}^q(X, \mathcal{O}_X) = H^{q+1}(\underbrace{K^\bullet \otimes \cdots \otimes K^\bullet}_{(n+1)\text{-times}})$$

Hence

$$H^q(X, \mathcal{O}_X) = \begin{cases} 0 & q \neq 0 \text{ or } n \\ k[x_0, ..., x_n] & q = 0 \\ k\text{-vector space generated by } \frac{1}{x_0^{m_0} \cdots x_n^{m_n}}, m_i > 0 & q = n \end{cases}$$

Corollary 25.11. $\mathbb{A}_k^n \setminus \{0\}$ is not affine for $n \geq 2$, because $H^n(X, \mathcal{O}_X) \neq 0$ for $n \geq 2$ and Serre's Theorem A and B.

25.4 §E. Calculation of Sheaf Cohomologies

(1) Grothendieck vanishing theorem

Theorem 25.12 (Grothendieck)**.** *Let X be a Noethrian topological space of dimension n, and let $\mathscr{F}$ be a sheaf of abelian groups. Then*

$$H^q(X, \mathscr{F}) = 0 \quad \textit{if } q > n$$

Proof. [Har77] , III, Theorem 2.7. □

Corollary 25.13. Let X be affine algebraic variety of dimension n. $\mathscr{F}$ is a quasi-coherent sheaf. Then

$$H^q(X, \mathscr{F}) = 0 \quad \text{if } q > n$$

(2) Finiteness

Theorem 25.14. *Let X be a projective algebraic variety, $\mathscr{F}$ a coherent sheaf on X. Then*

$$\dim_k H^q(X, \mathscr{F}) < +\infty, \quad \forall q \in \mathbb{Z}$$

Proof. Reduce to projective spaces. [Har77] , III, §5.2. □

Remark 25.15.

(1) If X is not projective, then the theorem does not hold, e.g. $H^0(\mathbb{A}_k^n, \mathcal{O}_{\mathbb{A}_k^n}) = k[x_1, ..., x_n]$.

(2) $\chi(t) = \sum_{q=0}^{n} (-1)^q \dim_k H^q(X, \mathscr{F}(t))$ is a polynomial in t, where $\mathscr{F}(t) = \mathscr{F} \otimes \mathcal{O}_X(t)$ and $t \in \mathbb{Z}$.

(3) Kodaira's vanishing Theorem

Theorem 25.16. *Let X be an irreducible nonsigular projective variety, $\mathscr{L}$ an ample line bundle on X. Then*

$$H^q(X, \omega_X \otimes \mathscr{L}) = 0, \quad \forall q \geq 1$$

where $\omega_X = \bigwedge^n \Omega_X$ is the canonical bundle of X, $n = \dim X$.

(4) Serre's duality

Theorem 25.17. *Let X be an irreducible nonsigular projective variety. Let $\mathscr{F}$ be a locally free coherent sheaf on X. Then there exists a natural isomorphism*

$$H^i(X, \mathscr{F}) \simeq H^{n-i}(X, \mathscr{F}^* \otimes \omega_X)$$

where $i \in \mathbb{Z}_{\geq 0}$ and $\mathscr{F}^* = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$ is the dual sheaf of $\mathscr{F}$.

Corollary 25.18 (Cohomologies of projective spaces)**.** Let $n \in Z_{>0}$. Let $S = k[x_0, ..., x_n]$ be the natural $\mathbb{Z}$-graded k-algebra.

(1) The natural morphism $S \to \bigoplus_{m \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ is an isomorphism of graded S-modules. In particular, we have

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \begin{cases} S_m & m \geq 0 \\ 0 & m < 0 \end{cases}$$

(2) $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$, $\forall 1 \leq i \leq n-1$, $\forall m \in \mathbb{Z}$.

(3) $H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)) = \begin{cases} S_{-m-n-1} & m \geq -n-1 \\ 0 & m > -n-1 \end{cases}$

Proof.

(1) $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \begin{cases} S_m & m \geq \\ 0 & m < 0 \end{cases}$ See [VII, §H].

(2) Recall that $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, hence

$$\begin{aligned} H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) &\simeq H^{n-i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)^* \otimes \omega_{\mathbb{P}^n}) \\ &\simeq H^{n-i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m) \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1)) \\ &\simeq H^{n-i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)) \end{aligned}$$

Hence, by Kodairas's vanishing theorem, if $-m > 0$, $n > i$, then $H^{n-i}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-m-n-1)) = 0$. Thus if $-m > 0$ and $n > i$, then

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0 \tag{1}$$

On the other hand,

$$\begin{aligned} H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) &\simeq H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m) \otimes \omega^*_{\mathbb{P}^n} \otimes \omega_{\mathbb{P}^n}) \\ & H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m+n+1) \otimes \omega_{\mathbb{P}^n}) \\ &= 0 \quad \text{if } m+n+1 > 0 \text{ and } i > 0 \end{aligned}$$

In all, $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$, $\forall 1 \le i \le n-1$, $\forall m \in \mathbb{Z}$. For $i = n$, using Serre's duality.

□

26 Lecture 26.

22/12/05.

(5) Direct sum

Proposition 26.1. Let $(\mathscr{F}_i)_{i \in I}$ be a family of quasi-coherent sheaves on affine algebraic variety X. Then we have a natural isomorphism for $\forall q \ge 0$.

$$H^q(X, \bigoplus_{i \in I} \mathscr{F}_i) \simeq \bigoplus_{i \in I} H^q(X, \mathscr{F}_i)$$

Proof. Take a suitable affine open covering of X and the use Cĕch cohomology or prove that cohomology commutes with direct limit[[Har77] , II, Proposition 2.9]. □

(6) Long exact sequence

Proposition 26.2. Let X be affine algebraic variety. Let $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$ be an exact sequence of coherent sheaves. Then we have a natural long exact sequence $i \in \mathbb{Z}$.

$$\cdots \longrightarrow H^i(X, \mathscr{F}) \longrightarrow H^i(X, \mathscr{E}) \longrightarrow H^i(X, \mathscr{G}) \longrightarrow H^{i+1}(X, \mathscr{F}) \longrightarrow \cdots$$

Corollary 26.3 (Theorem A and B)**.** Let X be an irreducible normal projective variety. Let $\mathscr{L}$ be an ample line bundle on X. $\mathscr{F}$ a coherent sheaf on X. Then there exists an integer $N > 0$ such that for $\forall m \ge N$, we have

1. $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ is globally generated.

2. $H^q(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0$, $\forall q \ge 1$.

Proof. Without loss of generality, we may assume that $\mathscr{L}$ is very ample. In particular, there exists $i : X \hookrightarrow \mathbb{P}^N$ an inclusion as a closed subset and $\mathscr{L} = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. Thus we may denote $\mathscr{L}$ by $\mathcal{O}_X(1)$.

On the other hand, as $\mathscr{F}$ is coherent and X is projective, there exists $n_i\mathbb{Z}$, $1 \geq i \geq s$ such that there exists a surjective morphism

$$\bigoplus_{i=1}^{s} \mathcal{O}_X(n_i) \twoheadrightarrow \mathscr{F} \quad \text{[Chap. VII. § E]}$$

□

Proof is left

Corollary 26.4. We define $h^i(X, \mathscr{F}) := \dim_k H^i(X, \mathscr{F})$. Then

$$h^i(\mathbb{P}^n, \mathrm{T}_{\mathbb{P}^n}) = \begin{cases} n^2 + 2n & i = 0 \\ 0 & i > 0 \end{cases}$$

Proof. Consider the Euler sequence:

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Taking dual yields:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n} \longrightarrow 0$$

Take long exact sequence:

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) & \longrightarrow & H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \longrightarrow \cdots \\ & \| & & & & \| & \\ & 0 & & & & 0 & \end{array}$$

For $i \geq 1$. Hence, we have $H^i(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) = 0$ for $i \geq 1$.

$$\begin{array}{ccccccc} 0 \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}) & \longrightarrow & H^i(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) & \longrightarrow & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \\ & \| & & & & & & \| \\ & k & & & & & & 0 \end{array}$$

$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}) \simeq \bigoplus H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \simeq S_1^{\oplus n+1} \Rightarrow h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}) = (n+1)^2$.

By the exact sequence, we have

$$\begin{aligned} h^0(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n}) &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1}) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \\ &= (n+1)^2 - 1 \\ &= n^2 + 2n \end{aligned}$$

□

Corollary 26.5 (Exercise)**.** $h^i(\mathbb{P}^n, \Omega_{\mathbb{P}^n}) = \begin{cases} 0 & i \neq 1 \\ 1 & i = 1 \end{cases}$

(7) Push-forward and pull-back

Proposition 26.6. Let $i : Y \hookrightarrow$ be a closed subvariety. Then for any coherent sheaves $\mathscr{F}$ on Y and $\mathscr{G}$ on X, we have

(i) $H^q(Y, \mathscr{F}) = H^q(X, i_*\mathscr{F})$, $\forall q \geq 0$.

(ii) $i^* i_* \mathscr{F} = \mathscr{F}$.

(iii) (projective formula) If $\mathscr{G}$ is locally free, then we have

$$i_*(\mathscr{F} \otimes i^*\mathscr{G}) = i_*\mathscr{F} \otimes \mathscr{G}$$

In particular, if $\mathscr{F} = \mathcal{O}_Y$, we get $i_* i^* \mathscr{G} = i_*\mathcal{O}_Y \otimes \mathscr{G}$.

Remark 26.7. Let $\mathscr{G}$ be a locally free sheaf on X. We want to compute $H^q(Y, i^*\mathscr{G})$. By (i)+(iii), we get

$$H^q(Y, {}^*\mathscr{G}) \simeq H^q(Y, {}_*\mathcal{O}_Y \otimes \mathscr{G})$$

Then we consider the following short exact sequence:

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0$$

Tensor it with $\mathscr{G}$ yields:

$$0 \longrightarrow \mathscr{I}_Y \otimes \mathscr{G} \longrightarrow \mathcal{O}_X \otimes \mathscr{G} \longrightarrow i_*\mathcal{O}_Y \otimes \mathscr{G} \longrightarrow 0$$

Corollary 26.8. Let $X \subseteq \mathbb{P}^{n+1}$ be a nonsingular hypersurface of degree d. Then

$$h^q(X, \mathcal{O}_X) = \begin{cases} 1 & q = 0 \\ 0 & 0 < q < n \\ h^{n+1}(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(-d)) & q = n+1 \end{cases}$$

In particular, we have $h^{n+1}(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(-d)) = h^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d-n-1))$.

Proof. Consider the short exact sequence:

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

where $i : X \hookrightarrow \mathbb{P}^{n+1}$. X is a prime Cartier divisor in $\mathbb{P}^{n+1}$ of degree d. Then

$$\mathscr{I}_X = \mathcal{O}_{\mathbb{P}^{n+1}}(-X) \simeq \mathcal{O}_{\mathbb{P}^{n+1}}(-d)$$

Hence, we get LEFT ☐

26.1 §F. Serre's GAGA Principle

$k = \mathbb{C}$, $X \subseteq \mathbb{P}^n(\mathbb{C})$ an irreducible normal projective variety.
X^{an} = analytic space associated to X, which means X(as sets)+ Euclidean topology on X i.e. $\mathbb{P}^n(\mathbb{C}) = \bigcup_{i=0}^n U_i$ where $U_i \simeq \mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$ with Euclidean topology on U_i is the same as that on $\mathbb{C}^n$.
X^{an} is a compact complex manifold.
$\mathcal{O}_{X^{\mathrm{an}}}$ = sheaf of holomorphic functions on X^{an}.
Then the identity map: $\mathrm{Id}^{\mathrm{an}} : (X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) \to (X, \mathcal{O}_X)$ is a morphism of locally ringed spaces.
Let $\mathscr{F}$ be a coherent sheaf on X. Then the analytic sheaf $\mathscr{F}^{\mathrm{an}}$ associated to $\mathscr{F}$ is defined as

$$\mathscr{F}^{\mathrm{an}} := (\mathrm{Id}^{\mathrm{an}})^* \mathscr{F} = (\mathrm{Id}^{\mathrm{an}})^{-1} \mathscr{F} \otimes_{(\mathrm{Id}^{\mathrm{an}})^{-1}\mathcal{O}_X} \mathcal{O}_X^{\mathrm{an}}$$

Fact 26.9.

(1) $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}}$ is an exact functor.

(2) $\mathscr{F}^{\mathrm{an}}$ is an analytic coherent sheaf. Ref: Demailly. Complex Analytic and Differential II, §3. IV. Sheaf Cohomology.

Theorem 26.10 (Serre, GAGA principle)**.** *Let X be an irreducible normal projective variety. Let X^{an} be the analytic space associated to X. For $\mathscr{F}$ a coherent sheaf on X, let $\mathscr{F}^{an}$ be the analytic sheaf associated to $\mathscr{F}$.*

(1) $\forall q \geq 0$, there exists a canonical isomorphism

$$H^q(X, \mathscr{F}) \simeq H^q(X^{an}, \mathscr{F}^{an}).$$

(2) If $\mathscr{F}$ and $\mathscr{G}$ are two coherent sheaves, then there exists a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \simeq \mathrm{Hom}_{\mathcal{O}_X^{an}}(\mathscr{F}^{an}, \mathscr{G}^{an}).$$

(3) For any analytic coherent sheaf on X^{an}, there exists a unique coherent sheaf $\mathscr{F}$ on X such that

$$\mathscr{F}^{an} \simeq \mathscr{E}.$$

Corollary 26.11 (Chow)**.** Let $X \subseteq \mathbb{P}^n(\mathbb{C})$ be a compact complex manifold. Then X is projective.

Proof. $\mathscr{I}_X^{\mathrm{an}} \hookrightarrow \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}$, where $\mathscr{I}_X^{\mathrm{an}}$ is the analytic ideal sheaf of X. By Oka's coherence Theorem, $\mathscr{I}_X^{\mathrm{an}}$ and $\mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}$ are coherent. By GAGA principle, both $\mathscr{I}_X^{\mathrm{an}}$ and $\mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}$ come from algebraic coherent sheaves, i.e. there exists

$$\begin{array}{ccc} \mathscr{I}_X & \hookrightarrow & \mathcal{O}_{\mathbb{P}^n} \\ \text{analytic}\downarrow & & \downarrow\text{analytic} \\ \mathscr{I}_X^{\mathrm{an}} & \hookrightarrow & \mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}} \end{array}$$

Hence, $X = \mathrm{Supp}(\mathcal{O}_{\mathbb{P}^{n,\mathrm{an}}}/\mathscr{I}_X^{\mathrm{an}}) = \mathrm{Supp}(\mathcal{O}_{\mathbb{P}^n}/\mathscr{I}_X)$ is algebraic. $\square$

27 Lecture 27.

22/12/07.

28 Lecture 28.

22/12/12. **(5) Second example of projective morphisms: blowing-up**
Construction I−(Blowing-up at a finite set of regular functions)
Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic variety. For some given regular functions $f_1, ..., f_r \in \Gamma(X, \mathcal{O}_X)$ on X, we set $U = X \setminus V(f_1, ..., f_r)$. As $f_1, ..., f_r$ then do not vanish simultaneously at any point of U, there is a well-defined morphism

$$f : U \to \mathbb{P}^{r-1}$$

by

$$x \mapsto [f_1(x) : \cdots : f_r(x)]$$

We consider its graph

$$\Gamma_f == \left\{(x, f(x)) \in U \times \mathbb{P}^{r-1} | x \in U\right\} \subseteq U \times \mathbb{P}^{r-1}$$

Then Γ_f is $\underline{\text{closed in } U \times \mathbb{P}^{r-1}}$, but in general $\underline{\text{NOT in } X \times \mathbb{P}^{r-1}}$. The closure of Γ_f in $X \times \mathbb{P}^{r-1}$ is called the **blowing-up of X at** $f_1, ..., f_r$. We usually denote it by $\widetilde{X} \subseteq X \times \mathbb{P}^{r-1}$ and there is a natural projective morphism $\pi : \widetilde{X} \to X$ to the first factor.

Remark 28.1 (Exceptional sets)**.**

(1) Let $f : X \to Y$ be a birational morphism. Let U be the largest open subset of Y such that $f^{-1}(U) \to U$ is an isomorphism. Then the **exceptional set** $\mathrm{Ex}(f)$ is defined as $X \setminus f^{-1}(U)$.

(2) Let $\pi : \widetilde{X} \to X$ be the blowing-up as Construction I. Then π induces an isomorphism $\Gamma_f \to U$. In particular, $\mathrm{Ex}(\pi) \subseteq \widetilde{X} \setminus \pi^{-1}(U)$.

(3) For $r = 1$, i.e. the blowingup of X at one regular function f, if X is irreducible and $f \neq 0$, the $\pi : \widetilde{X} \to X$ is an isomorphism:

$$\Gamma_f \overset{\text{open}}{\subseteq} X \times \mathbb{P}^0 = X$$

Which means $\overline{\Gamma_f}$ in $X \times \mathbb{P}^0 = X \times \mathbb{P}^0 = \widetilde{X}$ by irreducibility.

(4) The blowing-up $\widetilde{X} \subseteq X \times \mathbb{P}^{r-1}$ of X at $f_1, ..., f_r$ satisfies

$$\widetilde{X} \subseteq \left\{(x, y) \in X \times \mathbb{P}^{r-1} | y_i f_j(x) = y_j f_i(x), \quad \forall 1 \leq i, j \leq r\right\}$$

Example 28.2 (Blowing-up of $\mathbb{A}_k^n$ at coordinate functions)**.** Let $\pi : \widetilde{\mathbb{A}_k^n} \to \mathbb{A}_k^n$ be the blowing-up of $\mathbb{A}_k^n$ at $X_1, ..., X_n$. Then

$$\widetilde{\mathbb{A}_k^n} = \Big\{(x,y) \in \mathbb{A}_k^n \times \mathbb{P}^{n-1} | y_i x_j = y_j x_i \quad \forall 1 \leq i,j \leq n\Big\} := Y$$

Claim: $Y = \widetilde{\mathbb{A}_k^n}$.
Proof of the claim: Consider the affine open subset $U_i = \{(x,y) \in \mathbb{A} \times \mathbb{P}^{n-1} | y_1 = 1\}$. Then $Y \cap U_1 = \{X_j = X_1 Y_j | i = 2, ..., n\}$ because $Y_i X_j = Y_i X_1 Y_j = X_i Y_j$, $\forall i, j$. There exists an isomorphism:

$$\mathbb{A}_k^n \to Y \cap U_1$$

by

$$(x_1, y_2, ..., y_n) \mapsto (x_1, x_1 y_2, ..., x_1 y_n)[1 : y_2 : \cdots : y_n]$$

The same holds for the open subsets U_i of Y where $y_i = 1$, $i = 2, ..., n$. Hence Y is covered by $Y \cap U_i \simeq \mathbb{A}_k^n$ and $(1, ..., 1)[1 : \cdots : 1] \in \cap_{i=1}^n (Y \cap U_i)$
$\Rightarrow Y$ is irreducible and $Y = \widetilde{\mathbb{A}_k^n}$.

29 Lecture 29.

22/12/14.

Fact 29.1 (Blowing-up depend only on ideals)**.** The blowing-up of affine algebraic variety X at $f_1, ..., f_r \in A(X) = \Gamma(X, \mathcal{O}_X)$ depends only on ideal $I = \langle f_1, ..., f_r \rangle \subseteq A(X)$ i.e. if $f_1', ..., f_s' \in A(X)$ such that $\langle f_1', ..., f_s' \rangle = I$ and let $\pi : \widetilde{X} \to X$ and $\pi' : \widetilde{X'} \to X$ be the blowing-up of X at $f_1, ..., f_r$ and $f_1', ..., f_s'$ respectively. Then there exists a unique isomorphism $g : \widetilde{X} \to \widetilde{X'}$ fitting in the following commutative diagr[AM94]

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\exists! g} & \widetilde{X'} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & X & \end{array}$$

Construction II−(Blowing-up at ideals)

(a) Let X be affine algebraic variety. Let $I \subseteq A(X)$ be an ideal. The **blowing-up of** X **at** I is defined as the blowing-up of X at any finite set of generators of I.

(b) Let X be affine algebraic variety. Let $Z \subseteq X$ be a closed subvariety. The **blowing-up of** X **at** Z is defined as the blowing-up of X at $I(Z) \subseteq A(X)$. In this case, we also call Z the **centre of the blowing-up**.

Example 29.2.

(1) (Blowing-up of $\mathbb{A}_k^n$ at (0,...,0))

Let $\pi : \widetilde{\mathbb{A}_k^n} \to \mathbb{A}_k^n$ be the blowing-up of $\mathbb{A}_k^n$ at $(0,...,0)$. Then $\widetilde{A} = \{(x,y) \in \mathbb{A}_k^n \times \mathbb{P}^{n-1} | x_i y_j = x_j y_i, \quad 1 \leq i$
and $E = \pi^{-1}(0,...,0) = (0,...,0) \times \mathbb{P}^{n-1} \subseteq \widetilde{\mathbb{A}_k^n}$, where $\dim \mathbb{P}^{n-1} = n-1$ and $\dim \widetilde{\mathbb{A}_k^n} = n$. $\widetilde{\mathbb{A}_k^n} \cap \{Y_i \neq 0\} := U_i \simeq \mathbb{A}_k^n$, consider

$$\mathbb{A}_k^n \overset{\varphi_i}{\to} \widetilde{\mathbb{A}_k^n}]subseteq \mathbb{A}_k^n \times \mathbb{P}^{n-1} \overset{\mathrm{pr}_1}{\to} \mathbb{A}_k^n$$

by

$$(z_1, ..., z_n) \mapsto (z_1 z_i, ..., z_{i-1} z_i, z_i, z_{i+1} z_i, ..., z_n z_i)[z_1 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_n]$$

$E \cap U_i = \mathrm{pr_1}^{-1}(0,...,0) = \{z_i = 0\} \subseteq \mathbb{A}_k^n$. Geometrically, the blowing-up separates the lines passing through the origin by sending the point $x \in l$ to $(x, [l] \in \mathbb{P}(k^n) = \mathbb{P}^{n-1}) \in \mathbb{A}_k^n \times \mathbb{P}^{n-1}$.

(2)(blosing-up of $\mathbb{A}_k^n$ at a linear subspace)

Let $\pi : \widetilde{\mathbb{A}_k^n} \to \mathbb{A}_k^n$ be the blowing-up of $\mathbb{A}_k^n$ at $V(x_1, ..., x_r) \subseteq \mathbb{A}_k^n$. Then

$$\widetilde{\mathbb{A}_k^n} = \left\{(x,y) \in \mathbb{A}_k^n \times \mathbb{P}^{r-1} | x_i y_j = x_j y_i, \quad 1 \leq i,j \leq r\right\}$$

which means

$$\pi^{-1}(V(x_1,...,x_r)) = \underbrace{V(x_1,...,x_r) \times \mathbb{P}^{r-1}}_{\text{dim=n-1}} \subseteq \underbrace{\widetilde{\mathbb{A}_k^n}}_{\text{dim=n}}$$

Construction III−(general case)

Let X be a variety, $\mathscr{I} \subseteq \mathcal{O}_X$ an ideal sheaf. The **blowing-up** $\pi : \widetilde{X} \to X$ **of** X **at** $\mathscr{I}$ is a surjective proper birational morphism such that if $X = \cup U_i$ is an affine open covering and $\mathscr{I}|_{U_i} = \widetilde{I}_i$, where $I_i \subseteq A(U_i)$ is an ideal, then

$$\pi|_{\pi^{-1}(U_i):\pi^{-1}(U_i) \to U_i}$$

is the blowing-up of U_i at I_i.

Basic properties of blowing-up

Proposition 29.3.

(1) The blowing-up exists.

(2) The inverse image ideal sheaf $\mathscr{I}' = \pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\widetilde{X}}$ is an invertible sheaf and is π-ample. In particular, π is projective.

(3)

30 Lecture 30.

22/12/19.

31 Lecture 31.

22/12/21.

32 Lecture 32.

22/12/26.

References

[AM94] M.F. Atiyah and I.G. MacDonald. *Introduction To Commutative Algebra.* CRC Press, 1994.

[Har77] Robin Hartshorne. *Algebraic Geometry.* Springer, 1977.

[Mus] M. Mustaţă. *Introduction to algebraic geometry.* http://www-personal.umich.edu/ mmustata/ag0523.pdf.

[ZS75] Oscar Zariski and Pierre Samuel. *Commutative Algebra: Volume I.* Springer, 1975.